(1) [6 points]  
(a) Show that a divisible module over a PID is injective. 
(b) Let $R$ be a ring (not necessarily commutative). Prove that $\text{Hom}_R(R, G)$ is an injective (left) $R$-module for any divisible Abelian group $G$. 
(c) Use the fact “Every Abelian group can be embedded in a divisible abelian group” to prove that every (left) $R$-module can be embedded in an injective (left) $R$-module.

(2) [7 points] Let $R$ be an integral domain and let $K$ denote its quotient field. Prove: 
(a) $K$ is an injective $R$-module. 
(b) Every $K$-vector space is an injective $R$-module.

(3) [6 points] Let $R$ be a Noetherian ring and $M$ an $R$-module. 
Let $\text{Supp}(M) := \{p \in \text{Spec}(R) : M_p \neq 0\}$. 
(a) Let $x \in M$ and $p \in \text{Spec}(R)$. Show: $(Rx)_p \neq 0 \iff \text{Ann}(x) \subseteq p$. 
(b) Let $a \in R$ and $a_M : M \to M$, $x \to ax$. Prove: $a_M$ locally nilpotent $\iff a \in \bigcap_{p \in \text{Supp}(M)} p$ 
(c) Assume that $M$ is finitely generated. Prove: $\sqrt{\text{Ann}(M)} = \bigcap_{p \in \text{Supp}(M)} p$. 
(d) Apply (c) to deduce a well-known result on Nilradical of $R$.

(4) [6 points] Let $R$ be a commutative Artinian ring; that is, $R$ satisfies the descending chain condition (dcc). 
(a) Prove that $R$ satisfies the minimum condition; that is, every nonempty set of ideals of $R$ has a minimal element. 
(b) Prove that the nilradical of $R$ is nilpotent.

(5) [8 points] A commutative ring is quasi-Frobenius if it is Noetherian and injective as a module over itself. Let $K$ be a field. A (commutative) finite-dimensional $K$-algebra $R$ is called a Frobenius algebra if $R$ is isomorphic to its $K$-vector space dual $R^* = \text{Hom}_K(R, K)$ as $R$-modules. 
(a) Prove that every Frobenius algebra is quasi-Frobenius. 
(b) Let $R$ be a (commutative) finite-dimensional $K$-algebra. Prove: If there is $f \in R^*$ such that $\text{Ker}(f)$ contains no nonzero ideals, then $R$ is a Frobenius algebra. 
(c) Deduce from above: If $G$ is a finite (Abelian) group, then the group ring $K[G]$ is quasi-Frobenius.

(6) [6 points] Recall that a ring $R$ is semisimple if it is semisimple as an $R$-module. Prove that the following conditions are equivalent for a ring $R$: 
(i) $R$ is semisimple; 
(ii) Every (left) $R$-module is semisimple; 
(iii) Every (left) $R$-module is injective; 
(iv) Every (left) $R$-module is projective.

(7) [6 points] Let $R$ be a semisimple ring with $I_1, \ldots, I_s$ its non-isomorphic simple (left) ideals. Let $E$ be a nonzero $R$-module. Prove that $E = \bigoplus_{1 \leq i \leq s} E_i$ where $E_i = \text{Sum of all simple submodules of } E$ isomorphic to $I_i$ for $i = 1, \ldots, s$. 