Math 673, Exam 1, Semester 171
Duration: two hours

Q1) Consider the following integral equation: for $t \in D$,

$$\lambda x(t) - \mathcal{K}x(t) = y(t), \quad \text{with } \mathcal{K}x(t) = \int_D K(t,s)x(s)\,ds, \tag{1}$$

where $\lambda > 0$, D is a one-dimensional bounded closed domain and $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a compact operator ($\mathcal{X}$ is a Banach space).

a) Let $\mathcal{X} = C(D)$.

(i) For $\lambda = 40$, $K(t,s) = \mathrm{e}^{st}$ and $D = [0,2]$, show that problem (1) has a unique solution in $C(D)$ whenever $y \in C(D)$.

(ii) Define the approximate solution x_n of problem (1) using the degenerate kernel method.

(iii) How one can make sure that x_n converges to x? Then, determine the speed of convergence.

(iv) Define the degenerate kernel $\mathcal{K}_n$ such that the method is convergent when $\lambda = 2$, $K(t,s) = \mathrm{e}^{st}$ and $D = [0,1]$.

b) Let $\mathcal{X}_n \subset \mathcal{X}$ be a finite dimensional space with basis functions $\{\phi_1, \phi_2, \cdots, \phi_{d_n}\}$. Let $x_n \in \mathcal{X}_n$ be an approximate solution of problem (1).

(i) Define x_n using the continuous collocation method ($\mathcal{X} = C(D)$).

(ii) Define x_n using the Galerkin method ($\mathcal{X} = L^2(D)$).

c) Assume that $D = [a,b]$ and x_n is the piecewise linear Galerkin solution of problem (1) defined on a uniform mesh with step-size $h = (b-a)/n$, (the convergence rates of x_n to x is of order 2 provided that $x \in C^2[a,b]$). Define the iterated solution $\widehat{x}_n$. Show that under certain assumptions on λ, $\mathcal{K}$ and x, the convergence rates of $\widehat{x}_n$ to x is of order 4.

Q2) Consider the following problem: for $t \in [0, 1]$,

$$2x(t) - \mathcal{K}x(t) = y(t), \quad \text{with } \mathcal{K}x(t) = \int_0^1 \left(e^{st}\right) x(s)\, ds,$$

Let $t_j = j\delta$ $(0 \le j \le n)$ where $\delta = 1/n$, and let x_n be the associated piecewise-linear discontinuous collocation Gauss-Legendre solution.

(i) Define x_n.

(ii) Fix $n = 2$. Write the numerical scheme in a matrix form (Do not compute the integrals).

(iii) Define the iterated solution $\widehat{x}_n$, and state the expected convergence rates for smooth solution x and smooth kernel K, and sufficiently large n.

(iv) Find the operator $\mathcal{A}$ such that $x - \widehat{x}_n = \mathcal{A}(\mathcal{K} - \mathcal{K}\mathcal{P}_n)x$, where $\mathcal{P}_n$ is the Gauss-Legendre interpolatory projection operator, that is, $\mathcal{P}_n\phi = \phi$ at the composite Gauss-Legendre nodes.