*Final Exam for Math 673, Semester 171, Duration: 150 minutes*

Q1) Consider the following integral equation: for $t \in D := [0, T]$,

$$\lambda x(t) - \mathcal{K}x(t) = y(t), \quad \text{with } \mathcal{K}x(t) = \int_D K(t,s)x(s)\,ds, \tag{1}$$

where $\lambda > 0$, and the kernel $K(t,s)$ is continuous for $t$, $s \in D$.

(i) State the necessary conditions so that problem (1) has a unique solution in $C(D)$.

(ii) Define the Nystöm interpolation method using $n$-point Gauss-Legendre points on the interval $[0, T]$. Use the notations $w_j$ for the weights and $t_j$ for the nodes.

(iii) Write the scheme in a matrix form

(iv) State the sufficient conditions that guarantee the unique solvability of the Nyström scheme as well as its convergence. Discuss the speed of convergence.

(v) State explicitly how to compute $w_j$ and $t_j$ for $1 \le j \le n$ when $n = 2$ and $T = 1$.

vi) Define the piecewise linear Galerkin solution $x_n$ of problem (1) on a uniform mesh with a step-size $h = T/n$ in an abstract form, using an appropriate projection operator. Use $x_n$ to define a more efficient approximate solution.

Q2) Consider the following integro-differential equation: for $t \in [0, T]$,

$$u'(t) + a(t)u(t) + \int_0^t (t-s)^{-\alpha} u(s)\, ds = f(t), \quad \text{with } \; u(0) = 1, \tag{2}$$

where $0 < \alpha < 1$, and the functions $a$ and $f$ are smooth with $a(t) \geq a_0 > 0$ for all $t \in [0, T]$. Assume that problem (2) has a unique solution $u$ in the space $C^1[0, T]$.

(i) Verify that the integro-differential equation (2) is equivalent to a Volterra integral equation of a second kind that needs to be determined.

(ii) Define the discontinuous Galerkin solution $U \in \mathcal{W}_\tau$ of problem (2) over a mesh consists of $N$ subintervals of equal size $\tau$, with $U^0 = u(0) = 1$. Here $\mathcal{W}_\tau$ is the space of linear polynomials on each subinterval with one-side discontinuities at the nodes.

(iii) Determine the dimension of $\mathcal{W}_\tau$ and define explicitly a set of basis functions.

(iv) Show the existence and uniqueness of the DG solution $U$. Here, you may need to use the following non-negativity property:

$$\int_0^{t_n} \int_0^t (t-s)^{-\alpha} \phi(s)\, ds\, \phi(t)\, dt \geq 0.$$

(v) State the expected convergence rates of the DG solution (in the sup-norm) and the required smoothness assumptions on $u$ to observe it.

Q3) Consider the following integral equation: for $t \in D := [0, T]$,

$$u(t) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{u(s)}{\sqrt{t-s}}\, ds = 1, \tag{3}$$

Let $U \in \mathcal{W}_\tau$ be the piecewise-linear discontinuous collocation 2-point Gauss-Legendre solution of problem (3), defined on a uniform mesh with $n$ subintervals.

(i) Define $U$.

(ii) Show that the existence of $U$ follows from its uniqueness.

(iii) Show that the solution $U$ is unique on the first subinterval.