Final Exam for Math 690, Semester 171
Duration: 150 minutes

In this exam, $\Omega \subset \mathbb{R}^2$ is a bounded convex polygonal domain with boundary $\partial\Omega$, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product and $\|\cdot\|$ is the associated norm.

For $\alpha > 0$, $\mathcal{I}^\alpha$ is the RL time fractional integral of order α,

$$\mathcal{I}^\alpha \varphi(t) = \int_0^t \omega_\alpha(t-s)\varphi(s)\,ds \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0,$$

and D^α is the RL time fractional derivative,

$$\mathrm{D}^\alpha \varphi(t) := \frac{d^n}{dt^n}\mathcal{I}^{n-\alpha}\varphi(t), \quad n-1 < \alpha < n, \quad t > 0.$$

Q1. Mention briefly the main differences between the fractional and the classical derivatives.

Q2. Consider the following problem: for $0 < \alpha < 1$,

$$w'(t) + D^{1-\alpha}w(t) = \frac{1}{2}\sin(w(t)), \quad t \in (0,T], \quad \text{with } \ w(0) = 1.$$

Let $t_j = j\delta$ $(0 \le j \le N)$ where $\delta = T/N$. Let $W^j \approx w^j := w(t_j)$ (for $1 \le j \le N$) with $W^0 = 1$, be the implicit finite difference (backward Euler in time) solution, defined by

$$W^n - W^{n-1} + \int_{t_{n-1}}^{t_n} D^{1-\alpha}\overline{W}(t)\,dt = \frac{1}{2}\delta\,\sin(W^{n-1}), \quad \text{for } 1 \le n \le N,$$

where $\overline{W}(t) = W^i$ for $t_{i-1} < t \le t_i$, and for $1 \le i \le N$.

a) Show the existence of the finite difference solution W^n for $1 \le n \le N$.
b) Show that

$$|W^n| \le \delta \sum_{j=0}^{n-1} |W^j|, \quad \text{for } 1 \le n \le N.$$

c) Prove that $|W^n| \le \delta(1+\delta)^{n-1}$ for $1 \le n \le N$.
d) Determine whether the above finite difference scheme is unconditionally or conditionally stable. Justify your answer.

Q3. Consider the following problem

$$u'(x,t) - \mathrm{D}^{1-\alpha}\Delta u(x,t) + b\,u(x,t) = 0, \quad \text{in } \Omega\times(0,T],\ \ u(x,0) = v(x), \quad (1),$$

subject to homogeneous Dirichlet boundary conditions, where $0 < \alpha < 1$, b is a non-negative constant, and v is a given function.

a) Show that as α approaches 1, problem (1) reduces to the classical diffusion reaction problem.

Part A. For each $t \in (0,T]$, let $u_h(t)$ be the spatial (piecewise linear) continuous Galerkin finite element approximation of the exact solution $u(t)$ with $u_h(0) = v_h$ (approximates v). Use the notation S_h for the finite dimensional Galerkin solution.

i) Define $u_h(t)$ and then, show that $\|u_h(t)\| \le C\|v_h\|$ for $t > 0$.

ii) Choose $b = 0$ and v_h to be the L_2-projection of v on S_h. Let $\theta(t) = u_h(t) - R_h u(t)$ and $\rho(t) = u(t) - R_h u(t)$ where R_h is the Ritz projection on S_h. For any $\chi \in S_h$, show that

$$(t\theta(t),\chi) + \int_0^t \omega_\alpha(t-s)s(\nabla\theta(s),\nabla\chi)\,ds$$
$$= (t\rho(t),\chi) - \alpha\int_0^t \omega_{\alpha+1}(t-s)(\nabla\theta(s),\nabla\chi)\,ds.$$

iii) State the expected L_2-norm error from the spatial Galerkin approximation when $v \in L_2(\Omega)$.

Part B. Choose $b = 0$. Let $U \in \mathcal{W}_\delta$ be the semi-discrete piecewise-linear discontinuous Galerkin solution (in time) of problem (1). Here $\delta = T/N$ is the step-size of the uniform time mesh. Noting that $U^0 = v$.

i) Define the space $\mathcal{W}_\delta$ and the discontinuous Galerkin scheme.

ii) Show that U is unconditionally stable.

iii) Suppose that one will discretize the scheme in the above part (i) in space via the piecewise linear continuous Galerkin method over a quasi-uniform mesh. Let $\mathcal{W}_{\delta,h}$ be the space where we sought the approximate solution, denoted by U_h. Find the dimension of $\mathcal{W}_{\delta,h}$.

iv) Show the existence and uniqueness of the numerical solution U_h.