1. [10pts] Let \( P, Q, R \) be statements. Is the logical equivalence

\[
P \implies (Q \implies R) \equiv (P \land Q) \implies R
\]

true? Justify.

**Solution.** We can use a truth table to show that the logical equivalence is true. A shorter way is as follows.

\[
P \implies (Q \implies R) \equiv P \lor (\neg Q \lor R) \equiv (P \lor \neg Q) \lor R \equiv (P \land Q) \lor R \equiv (P \land Q) \implies R
\]

2. [10pts] Let \( A, B, C \) be sets, \( A \neq \emptyset \).

(a) Suppose \( A \cap B = A \cap C \). Is it true that \( B = C \)? Justify.

**Solution.** No, take \( A = B = \{0\}, C = \{0, 1\} \), then \( A \cap B = A \cap C = \{0\} \) but \( B \neq C \).

(b) Suppose \( A \times B = A \times C \). Prove that \( B = C \).

**Proof.** We first prove that \( B \subseteq C \). Let \( b \in B \). Since \( A \neq \emptyset \), we can choose an element \( a \in A \). We have \((a, b) \in A \times B \), so \((a, b) \in A \times C \), i.e. \( b \in C \). Similarly, \( C \subseteq B \). Hence \( B = C \). ■

3. [10pts] (a) Find subsets \( X, Y \) of \( \{1, 2, 3\} \) such that \( \mathcal{P}(X \cup Y) \leq \mathcal{P}(X) \cup \mathcal{P}(Y) \).

**Solution.** Take \( X = \{1\} \), \( Y = \{2\} \). Then \( \{1, 2\} \in \mathcal{P}(X \cup Y) \) but \( \{1, 2\} \notin \mathcal{P}(X) \) (since \( 2 \notin X \)) and \( \{1, 2\} \notin \mathcal{P}(Y) \) (since \( 1 \notin Y \)).

(b) Let \( A, B \) be sets. Prove that \( \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \).

**Proof.** Let \( S \in \mathcal{P}(A) \cup \mathcal{P}(B) \). Suppose first that \( S \in \mathcal{P}(A) \), then \( S \subseteq A \subseteq A \cup B \), so that \( S \in \mathcal{P}(A \cup B) \).

Similarly, if \( S \in \mathcal{P}(B) \), we obtain \( S \in \mathcal{P}(A \cup B) \). This proves \( \mathcal{P}(A) \) and \( \mathcal{P}(B) \) are both subsets of \( \mathcal{P}(A \cup B) \) so that \( \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) \). ■

4. [10pts] Let \( x, y \) be real numbers.

(a) Using properties of absolute value, prove that \( |x + y| - |x - y| \leq 2|x| \).

**Proof.** \( 2|x| + |x - y| = |2x| + |y - x| \geq |2x + (y - x)| = |x + y| \). ■

(b) Prove that if \( |x| < |y| \), then \( x^2 - 2xy + 3y^2 \geq 0 \). Is it true that \( x^2 - 2xy + 3y^2 > 0 \)? Justify.

**Solution.**

- **Proof.** We have \( x^2 - 2xy + 3y^2 = (x - y)^2 + 2y^2 \geq 0 \) for all real numbers. Hence the statement to prove is trivially true. ■

- It is true that \( x^2 - 2xy + 3y^2 > 0 \) when \( |x| < |y| \): This is because \( |x| < |y| \) implies \( x \neq y \) i.e. \( (x - y)^2 > 0 \), hence \( (x - y)^2 + 2y^2 > 0 \) i.e. \( x^2 - 2xy + 3y^2 > 0 \).

[Note that \( |x| < |y| \) also implies that \( y^2 > 0 \) and also that if we do not assume \( |x| < |y| \), then \( x^2 - 2xy + 3y^2 > 0 \) is not always true: take \( x = y = 0 \).]
5. [10pts] Let $x, y$ be integers.

(a) Prove that if $4 \mid (x^2 + y^2)$, then $x$ and $y$ are even.

**Proof.** We use the contrapositive and prove that if $x$ and $y$ have different parity or if both are odd, then $x^2 + y^2$ is not divisible by 4.
If $x$ and $y$ have different parity, then $x^2$ and $y^2$ also have different parity and so $x^2 + y^2$ is odd and cannot be divisible by 4.
If $x$ and $y$ are odd, then there are integers $h, k$ such that $x = 2h + 1, y = 2k + 1$. In this case, $x^2 + y^2 = 4(k^2 + k + h^2 + h) = 2$, which is not divisible by 4.

(b) Prove that if $3 \mid 5x$, then $3 \mid x$.

**Proof.** If $3 \mid 5x$, then $3 \mid (6x - 5x)$ (because $3 \mid 6x$), i.e. $3 \mid x$.

[Note. This problem can also be solved using congruences mod 4 in (a) and mod 3 in (b).]

6. [10pts] Let $a, b \in \mathbb{Z}$.

(a) Prove that if $a \equiv 2 \pmod{6}$ and $b \equiv 1 \pmod{4}$, then $2a - 3b \equiv 1 \pmod{12}$.

**Proof.** If $a \equiv 2 \pmod{6}$ and $b \equiv 1 \pmod{4}$, then $2a \equiv 4 \pmod{12}$ and $3b \equiv 3 \pmod{12}$. Hence $2a - 3b \equiv 1 \pmod{12}$.

(b) Prove that $5 \nmid a$ if and only if $a^4 \equiv 1 \pmod{5}$.

**Proof.** Suppose $5 \nmid a$. Then $a \equiv \pm 1 \pmod{5}$ or $a \equiv \pm 2 \pmod{5}$. In the first case, $a^4 \equiv 1 \pmod{5}$; and in the second case, $a^4 \equiv 16 \equiv 1 \pmod{5}$.
Conversely, if $a^4 \equiv 1 \pmod{5}$, then clearly $a$ is not divisible by 5 (otherwise, $a \equiv 0 \pmod{5}$ and then $a^4 \equiv 0 \pmod{5}$).