1. [10pts] Let $R$ be a relation defined on $\mathbb{R}$ by $xRy$ iff $xy \geq 0$. Is the relation $R$

(i) reflexive? (ii) symmetric? (iii) transitive? Justify your answers.

**Solution.** (i) Let $x \in \mathbb{R}$. Then $x^2 \geq 0$, i.e. $xRx$. Hence $R$ is reflexive.

(ii) Let $x, y \in \mathbb{R}$ such that $xRy$. Then $yx = xy \geq 0$, i.e. $yRx$. Hence $R$ is symmetric.

(iii) Let $x = 1, y = 0, z = -1$. Then $xRy$ and $yRz$ (because $xy \geq 0$ and $yz \geq 0$), but $xz = -1 \nexists 0$.

Hence $R$ is not transitive.

2. [10pts] Prove that the function $f : \mathbb{Z}_{31} \rightarrow \mathbb{Z}_{31}$ given by $f ([x]) = [1 + 2x]$ is a bijection.

**Proof.** Observe first that the domain and codomain of the function $f$ are finite ($|\mathbb{Z}_{31}| = 31$) and they have the same cardinality (in fact they are equal), hence $f$ is one-to-one iff it is onto. This means that to prove that $f$ is bijective, it is enough to prove that it is one-to-one. For that, let $x, y \in \mathbb{Z}$ be such that $f ([x]) = f ([y])$. Then $[1 + 2x] = [1 + 2y]$ i.e. $1 + 2x \equiv 1 + 2y \pmod{31}$. We thus get $2x \equiv 2y \pmod{31}$ and hence $16 \times 2x \equiv 16 \times 2y \pmod{31}$. Since $32 \equiv 1 \pmod{31}$, we obtain $x \equiv y \pmod{31}$ i.e. $[x] = [y]$. This proves that $f$ is one-to-one. ■