Exercise 1 (10 points).

Use the augmented matrix and reduced echelon form to solve the following system.

\[
\begin{align*}
\begin{bmatrix}
  x - y + z &= 1 \\
  2x + y - z &= -1 \\
  2x + 4y + 14z &= 8
\end{bmatrix}
\end{align*}
\]
Exercise 2 (10 points).

(1) Under which conditions on $a$ and $b$, the system (S) below has a unique solution.
\[
\begin{align*}
    x - y + z & = 1 \\
    2x + y - az & = -1 \\
    2x + 4y + 14bz & = 1
\end{align*}
\]

(2) Without any calculations show that the system ($S_1$) has no solution.
\[
\begin{align*}
    x - y + z & = 1 \\
    2x + y + 15z & = -1 \\
    2x + 4y + 28z & = 1
\end{align*}
\]
Exercise 3 (15 points).
Recall that a square matrix $A$ is symmetric if $A = A^T$ and skew symmetric if $A^T = -A$.
(1) Let $A$ by any square matrix. Prove that $A = M - N$ where $M$ is symmetric
and $N$ is skew-symmetric.

(2) Application: Write the matrix
\[
\begin{pmatrix}
1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 \\
3 & 3 & 4 & 4 \\
4 & 4 & 3 & 3 \\
\end{pmatrix}
\]
as a difference of symmetric and skew-symmetric matrices.
Exercise 4 (10 points).

Use Gauss-Jordan Method to find the inverse (if it exists) of the matrix

\[
\begin{pmatrix}
2 & 1 & 2 \\
1 & 2 & 1 \\
3 & 1 & 2
\end{pmatrix}
\]
Exercise 5 (15 points).

Use Cramer method to solve the system (S):

\[
\begin{pmatrix}
    x + 3y + z &= 1 \\
    2x + y + z &= 5 \\
    -2x + 2y - z &= -8
\end{pmatrix}
\]
Exercise 6 (15 points).

Let \( N = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \) and \( A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \).

(1) Find \( N^2 \), \( N^3 \) and \( N^n \) for every \( n \geq 3 \).

(2) Use question (1) to find \( A^n \) for every \( n \geq 2 \) [Hint: Write \( A = N + I \)].

(3) Use question (2) to find \( A^{101} \).
Exercise 7 (10 points).

Let \( A = \begin{pmatrix} 2 & 3 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 0 \end{pmatrix} \).

(1) Find the adjoint matrix \( \text{adj}(A) \) of \( A \).
(2) Find the inverse of the matrix \( A \) using its adjoint.
Exercise 8 (15 points).
(2) Find all $3 \times 3$ matrices $A$ such that $AB = BA$ for every $3 \times 3$ matrix $B$. 