Exercise 1 (10 points, 5-5).
Determine whether the given vectors span \( \mathbb{R}^3 \) or not. Justify.
(1) \( v_1 = (1, 2, 1), v_2 = (1, 0, 1), v_3 = (2, 0, 1) \).
(2) \( v_1 = (0, 2, 1), v_2 = (1, 1, 1), v_3 = (1, 5, 3), v_4 = (2, 4, 3) \).
Exercise 2 (10 points, 5-5).
Which one of the following subsets is a subspace of the corresponding vector space.

1) $W = \{(a, b, c) \in \mathbb{R}^3 | a + b = 2c\}.$
2) $W = \{(a, b, c) \in \mathbb{R}^3 | a^2 + b^2 + c^2 \leq 1\}.$
Exercise 3 (20 points, 5-5-5-5).

Let $V = \mathbb{R}$ be the vector space over the field $\mathbb{Q}$ of rational numbers and $W := \{a + b\sqrt{2} + c\sqrt{3} | a, b, c \in \mathbb{Q}\}$.

(1) Prove that $W$ is a subspace of $V$.

(2) Prove that $\{1, \sqrt{2}, \sqrt{3}\}$ are linearly independent.

(3) Find a basis and the dimension of $W$, $\dim_\mathbb{Q}(W)$.

(4) Find a infinite set of linearly independent vectors of $V$. 

**Exercise 4** (15 points, 5-5-5).
Let $V = \mathbb{P}_2 := \{ f \in \mathbb{R}[X] | \deg(f) \leq 2 \}$, $S = \{ 1, X, X^2 \}$ the standard basis of $V$ and $B = \{ 1 + X, 1 + X^2, 1 + X + X^2 \}$.

1. Prove that $B$ is a basis for $V$.
2. Find the transition matrix $P$ from $B$ to $S$.
3. Find the coordinate vector of $f = 4 + 3X + 7X^2$ in the basis $B$. 

Exercise 5 (15 points, 5-5-5).
Let $V$ be a vector space over a field $\mathbb{F}$, $B_1, B_2, B_3$ three bases of $V$, $P$ the transition matrix from $B_2$ to $B_1$ and $Q$ the transition matrix from $B_3$ to $B_2$.
(1) Express the transition matrix $N$ from $B_3$ to $B_1$ in terms of $P$ and $Q$.
(2) Express the transition matrix $M$ from $B_1$ to $B_3$ in terms of $P$ and $Q$.
(3) Assume that $V = \mathbb{R}^3$, $P = \begin{pmatrix} 2 & 3 & 2 \\ 2 & -1 & 4 \\ 3 & 2 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $[x]_{B_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Express $[x]_{B_1}$.
Exercise 6 (15 points, 5-5-5).

(1) Let $A$ be an $n \times n$ matrix. Prove that $A$ is invertible if and only if $\text{rank}(A) = n$.

Let $A = \begin{pmatrix} 2 & 3 & 2 & 4 \\ 2 & -1 & 4 & 4 \\ 3 & 2 & 0 & 6 \\ 1 & 0 & 1 & 2 \end{pmatrix}$.

(2) Find a basis and the dimension of the row space of $A$.

(3) Find a basis and the dimension of the row space of $A$. 
Exercise 7 (15 points, 5-5-5-5).
Let $V = \mathbb{R}^3$ and $T : V \rightarrow V$ defined by $T(a, b, c) = (b + c, a + c, a + b)$.
(1) Verify that $T$ is a linear transformation.
(2) Find $\ker(T)$, $\dim(\ker(T))$, $\operatorname{Im}(T)$ and $\dim(\operatorname{Im}(T))$.
(3) Find the matrix representing $T$ is the standard basis $B$ of $V$. 