Exercise 1 (20 points, 5-5-5-5).

Consider a non-homogeneous system (S): \( AX = Y \), where \( A \) is an \( n \times n \) matrix.

1. Prove that if \( A \) is invertible, then the system has exactly one solution.

2. Assume that \( X_1, X_2 \) are two solutions of (S) and \( \alpha_1, \alpha_2 \) real numbers. Under which condition \( \alpha_1 X_1 + \alpha_2 X_2 \) is a solution of (S).

3. Verify that \( X_1 = (-1, 0, 1) \) and \( X_2 = (0, -1, 0) \) are solutions of the system (S)
\[
\begin{pmatrix}
2x + y + z &= -1 \\
x - y + 2z &= 1 \\
x + 2y - z &= -2
\end{pmatrix}
\]

4. Without any calculations, justify why \( X_1 + X_2 \) is not a solution while \( \frac{1}{3} X_1 + \frac{2}{3} X_2 \) is a solution of (S). [Do not solve the system].
**Exercise 2** (20 points, 5-5-5-5).

Let $V = \mathbb{R}^3$ endowed with the standard inner product $W = \{ (a, b, c) \in \mathbb{R}^3 | a + b - 2c = 0 \}$ and $F = \{ (a, b, c) \in \mathbb{R}^3 | a - b - 2c = 0 \}$.

1. Which one of the subsets $W, F$ is a subspace of $V$?
2. Is $W \cup F$ a subspace of $V$?
3. Are $W$ and $F$ orthogonal?
4. Find the orthogonal subspace $W^T$ of $W$. 


Exercise 3 (20 points, 5-5-5-5).
Let \( V = M_2(\mathbb{R}) \) be the vector space of \( 2 \times 2 \) real matrices, \( A \) a \( 2 \times 2 \) fixed matrix and \( T : V \to V \) defined by \( T(B) = AB \) for every \( B \in V \).

(1) Prove that \( T \) is a linear transformation.

(2) Prove that if \( A \) is invertible, then \( T \) has an inverse, that is, \( T^{-1} : V \to V \) such that \( T \circ T^{-1} = T^{-1} \circ T = id \) where \( id : V \to V \) is the identity map.

(3) Put \( A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) and let \( S = \{M_1, M_2, M_3, M_4\} \) be the standard basis of \( V \).

Find \( [T]_S \).

(4) Find \( [T^{-1}]_S \).
Exercise 4 (20 points, 5-5-5-5).
Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ the standard basis of $V$ and $T : V \to V$ defined by $T(a, b, c) = (a + b + 2c, b, b + 3c)$.

(1) Find the matrix $A = [T]_S$ representing $T$ is the basis $S$.
(2) Prove that $A$ is diagonalizable.
(3) Find a matrix $P$ such that $P^{-1}AP$ is diagonal.
(4) Find a basis $B$ of $V$ such that the matrix $[T]_B$ is diagonal.
**Exercise 5** (20 points, 5-5-5-5).

Let $A$ be a $3 \times 3$ matrix and $f(X) = X^3 + a_2X^2 + a_1X + a_0$ its characteristic polynomial.

1. Prove that if $a_0 \neq 0$, then $A$ is invertible and find its inverse.
2. If $a_0 = 0$, is $A$ necessarily invertible? Justify by an example.
3. Suppose that $A$ is diagonalizable and $D$ its diagonal matrix is invertible. Prove that $A$ is invertible.
4. Assume that all eigenvalues of $A$ are equal to 1 or $-1$. Prove that $A = A^{-1}$. 
Exercise 6 (20 points, 5-5-5-5).
Let \( V = C([0, 1]) \) be the vector space of all continuous functions on \([0, 1]\) endowed with the inner product \( (f|g) = \int_0^1 f(x)g(x)dx \).
(1) Determine the angle \( \theta \) between \( 1 \) and \( x \).
(2) Find the vector projection \( p \) of \( 1 \) onto \( x \).
(3) Verify that \( 1 - p \) is orthogonal to \( p \).
(4) Compute \( ||1 - p||, \|p\| \text{ and } ||1|| \) and verify Pythagore’s law.
Exercise 7 (20 points, 4-7-9).
Let \( p_0(x), p_1(x) \) and \( p_2(x) \) orthogonal with respect to the inner product
\[
(p(x)|q(x)) = \int_0^1 \frac{p(x)q(x)}{1+x^2} \, dx.
\]
Find \( p_0(X), p_1(x) \) and \( p_2 \) if all polynomials have leading coefficient equal to 1.
Exercise 8 (20 points, 6-8-3-3).
Consider the quadratic form \( q \) of \( \mathbb{R}^3 \) defined by \( q(x, y, z) = 3x^2 - 3z^2 + 8yz \).

1. Write \( q \) in the matrix form and find the eigenvalues of its matrix.
2. Find a the canonical quadratic form associated to \( q \).
3. Find the signature of \( q \).
4. Find the rank of \( q \).
Exercise 9 (20 points, 5-5-5-5).
Consider the quadratic equation $5x^2 + 5y^2 - 6xy - 24\sqrt{2}x + 8\sqrt{2}y + 56 = 0$.

(1) Write the equation in the matrix form and find the eigenvalues and eigenvectors of its matrix $A$.
(2) Find an orthogonal matrix $P$ and use the substitution $X = PX'$ to transform the equation to a simple form.
(3) Identify the obtained new equation with its rotation/translation axes.
(4) Sketch the graph of the equation.
Exercise 10 (20 points, 6-8-6).
Consider the function $F(x, y) = 3x^2 - xy + y^2$.
(1) Find all stationary points of $F(x, y)$.
(2) Find the Hessian matrix $H(X_0)$ and its eigenvalues for each stationary point $X_0$.
(3) Classify the stationary points (local max, local min, and saddle)