(1) (a) A collection $C$ of subsets of $X$ is an algebra with the following property: if $E_n \in C$ and $E_n \subseteq E_{n+1}$, $n = 1, 2, \cdots$, then $\bigcup_{n=1}^{\infty} E_n \in C$. Prove that $C$ is a $\sigma$-algebra.

(b) Let $f : [a, b] \to [-\infty, \infty]$ be a measurable function. Suppose that $f$ takes the value $\pm \infty$ only on a set of (Lebesgue) measure zero. Prove that for any $\varepsilon > 0$ there is a positive number $M$ such that $|f| < M$, except on a set of measure less than $\varepsilon$. 
(2) (a) Assume that $m^*$ is an outer measure on $2^X$ and pick $A \subseteq C \subseteq X$. Show that if a set $B$ is $m^*$-measurable and satisfies $A \subseteq B$ and $m^*(A) = m^*(B)$, then $m^*(C) = m^*(B \cup C)$.

(b) Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on a set $E \subset \mathbb{R}$, where $E$ is of finite Lebesgue measure ($m(E) < \infty$). Suppose that there is $M > 0$ such that $|f_n(x)| \leq M$ for $n \geq 1$ and for all $x \in E$, and suppose that $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \in E$. Use Egoroff’s theorem to prove that

$$\int_E f(x) \, dx = \lim_{n \to \infty} \int_E f_n(x) \, dx.$$
(3) Let $f(x)$ be a real-valued Lebesgue integrable function on $[0, 1]$.

(a) Prove that if $f > 0$ on a set $E \subset [0, 1]$ of positive measure, then

$$\int_{E} f(x) \, dx > 0.$$ 

(b) Prove that if

$$\int_{[0,t]} f(x) \, dx = 0, \text{ for each } t \in [0, 1],$$

then $f(t) = 0$ for almost all $t \in [0, 1]$. 
(4) (a) Suppose \( f : \mathbb{R} \to \mathbb{R} \) is measurable. Use the definition of measurability to prove that for every integer \( n \geq 2 \), the function \( f^n \) defined by \((f^n)(x) = (f(x))^n \) is also measurable. [Note: In your proof you should not use the fact that the product of measurable functions is measurable.]

(b) Set \( f_n(x) = \frac{n + \cos(nx)}{2n + 1} \), for \( x \in (0, +\infty) \) and \( n \in \mathbb{N} \). Evaluate with proof \( \lim_{n \to \infty} \int_0^n f_n(x) dx \).
(5) (a) Show that the monotone convergence theorem may not hold for decreasing sequences of functions.

(b) Let $Q = [0, 1] \times [0, 1] = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $f : Q \to \mathbb{R}$. Assume that $x \to f(x, y)$ is a measurable function for each fixed value of $y \in [0, 1]$ and the partial derivative $\frac{\partial f}{\partial y}$ exists. Suppose there is a function $g$ that is integrable over $[0, 1]$ such that

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq g(y), \text{ for all } (x, y) \in Q.$$ 

Prove that

$$\frac{d}{dy} \left[ \int_0^1 f(x, y) \, dx \right] = \int_0^1 \frac{\partial f}{\partial y}(x, y) \, dx, \text{ for all } y \in [0, 1].$$