Department of Mathematics and Statistics, KFUPM
Midterm Exam, Math 571, 07 March, 2018, Duration: 2 hours

P. 1. Consider the IVP: $y' = \cos(y) + \sin(x)$ for $x \in [x_0, x_M]$ with $y(x_0) = 0$.
a) Show that this IVP has a unique solution $y \in C^1[x_0, b]$ for some $b > 0$.
b) Define the one-step explicit and implicit Euler methods of the above IVP.
c) Show that the global error of the explicit Euler scheme is $O(h)$.

Consider the IVP:

$$y'(x) = f(x,y), \quad \text{for } x \in [x_0, x_M], \quad \text{with } y(x_0) = y_0. \qquad (1)$$

In all questions below, $f(.,.)$ is smooth and is assumed to satisfy a Lipschitz condition in the second variable. The approximate solution of $y^n := y(x_n)$ is denoted by y_n. Let $f_n := f(x_n, y_n)$, $f^n := f(x_n, y^n)$ for $0 \le n \le M$ where M is the number of mesh elements, and $h = x_n - x_{n-1}$.

P. 2. Consider the following implicit scheme

$$y_{n+1} - y_n = \frac{h}{2}\big(f_{n+1} + f_n\big), \quad n \ge 0.$$

Given that the truncation error

$$T^n := \frac{y^{n+1} - y^n}{h} - \frac{1}{4}[f^{n+1} + 3f^n],$$

is of order h^2. Show that

$$\max_{1 \le n \le M} |y^n - y_n| \le C h^2.$$

P. 3. State the general form of the three-stage RK method for the numerical solution of the IVP (1). Give an example of a consistent $O(h^3)$ accurate three-stage RK method (Justify your answer).

P. 4. Sketch four different rooted trees of order six (one root plus five vertices), and for each one, write the corresponding polynomial in the coefficients of the 6-stage RK method.

P. 5. Find all non-negative real values of δ such that the linear two-step method

$$y_{n+2} - \delta\, y_{n+1} - y_n = \frac{h}{3}(f_{n+2} + 4f_{n+1} + f_n) \quad \text{is zero} - \text{stable}.$$

P. 6. Consider the following two-stage RK method

$$y_{n+1} = y_n + \frac{h}{2}[f_n + f(x_{n+1}, y_n + hf_n)].$$

Show that the truncation error is of order h^2. Determine the interval of absolute stability.