Chapter 4 Duality

Primal
\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

Dual
\[
\begin{align*}
\text{max} & \quad \lambda^T b \\
\text{s.t.} & \quad \lambda^T A \leq c\\
& \quad \lambda \geq 0
\end{align*}
\]

Duality theorem
Lemma 1. If \( x \) and \( \lambda \) are feasible for (3) and (4), respectively, then \( c^T x \geq \lambda^T b \).

If either of (3) or (4) has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has unbounded objective, the other problem has no feasible solution.

For (3) suppose that \( x = (x_B, x_N) \) with basis \( B \) is a feasible basic optimal solution.

We know that \( x_B = \bar{B} \bar{b} \) and reduced cost vector is
\[
\bar{r}_B = \bar{c}_B - \bar{B} \bar{B}^{-1} D
\]
\[
\bar{r}_N = \bar{c}_N - \bar{B} \bar{B}^{-1} \bar{N}
\]

Since \( \bar{r}_N \geq 0 \rightarrow \bar{c}_B^t N \leq \bar{c}_N^t \).

Let us suppose that \( \lambda = \bar{c}_B^t \rightarrow \lambda^T A = [\lambda_B, \lambda_N] = [\bar{c}_B^t, \bar{c}_B^t \bar{B}^{-1} \bar{N}] \leq [\bar{c}_B^t, \bar{c}_B^t] = \bar{c}^t \rightarrow \lambda^T A \leq \bar{c}^t \) so \( \lambda \) is feasible for the dual.

and \( \lambda^T B = [\bar{c}_B^t] \bar{B}^t \bar{b} = \bar{B}^t \bar{B} \bar{b} \rightarrow \) this establishes the optimality of \( \lambda \) for its dual.
Duality

Primal
obj max
\[
\begin{align*}
\text{constraint} & \leq 0 \\
\text{constraint} & \geq 0 \\
\text{positive variable} & = 0 \\
\text{negative variable} & = 0 \\
\text{free variable} & = 0 \\
\end{align*}
\]

Ex

Max
\[
\begin{align*}
2x + 3y + 5z \\
\text{s.t.} \quad & x + 2y + z \leq 3 \\
& 3x + y + z \leq 4 \\
& y + y + 2z \leq 5 \\
& x, y, z \geq 0 \\
\end{align*}
\]

Dual
obj min
\[
\begin{align*}
\text{positive variable} & = 0 \\
\text{negative variable} & = 0 \\
\text{free variable} & = 0 \\
\text{constraint} & \leq 0 \\
\text{constraint} & \geq 0 \\
\end{align*}
\]

Ex 1:

Max
\[
\begin{align*}
2x + y \\
\text{s.t.} \quad & x + 3y \leq 6 \\
& 2x + y \leq 4 \\
& x, y \geq 0 \\
\end{align*}
\]

\[
\text{min } 6x + u \beta \\
\text{s.t.} \quad & x + 2\beta \geq 2 \\
& 3\alpha + \beta \geq 1 \\
& \alpha, \beta \geq 0 \\
\]
\[ \text{max } 2x + y \]
\[ \text{s.t. } \]
\[ x + 3y \leq 5 \Rightarrow x \]
\[ 2x + y \leq 8 \Rightarrow y \]
\[ x, y \geq 0 \]
\[ z = 3y + x \Rightarrow x = 5 - 3y \]
\[ 2x + y = 4 \quad (y = 0) \]

- What happens if the rhs of const 1 is reduced to 5?
- What happens if the rhs of const 2 is increased by 2?
- What happens if the objective of \( y \) is increased?

What can you say about the dual optimal solution?