Math201.01, Quiz #3, Term 173

Name: Solutions

ID #: **Serial #:**

1. [5 points] Find the local max/min values and saddle points of

$$f(x,y) = x^2 - 4x + xy^2.$$

2. [5 points] Find the extreme values of

$$f(x,y,z) = (x+1)^2 + (y-2)^2 + (z-1)^2$$

subject to the constraint $x^2 + y^2 + z^2 = 6$.

Good luck,

Ibrahim Al-Rasasi

[1] $f_x(x,y) = 2x - 4 + y^2$; $f_y(x,y) = 2xy$

$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 2x - 4 + y^2 = 0 & \text{—— (1)} \\ 2xy = 0 & \text{—— (2)} \end{cases}$ 1

(2) $\Rightarrow x = 0$ or $y = 0$

- $x = 0 \overset{(1)}{\Longrightarrow} -4 + y^2 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2 \Rightarrow \boxed{(x,y) = (0,-2),\ (0,2)}$
- $y = 0 \overset{(1)}{\Longrightarrow} 2x - 4 = 0 \Rightarrow x = 2 \Rightarrow \boxed{(x,y) = (2,0)}$ 1.5

- f_x & f_y exist at all pts (x,y).
- Test: $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2x$; $f_{xy}(x,y) = 2y$ 1

$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 2(2x) - (2y)^2 = 4x - 4y^2 = 4(x - y^2)$

- $(0,-2)$: $D(0,-2) = 4(0-4) = -16 < 0 \Rightarrow f$ has a saddle pt at $(0,-2)$
- $(0,2)$: $D(0,2) = 4(0-4) = -16 < 0 \Rightarrow$ ―――― $(0,2)$ 1.5
- $(2,0)$: $D(2,0) = 4(2-0) = 8 > 0$ & $f_{xx}(2,0) = 2 > 0$

$\Rightarrow f$ has a local min at $(2,0)$.

- The local min value is $f(2,0) = 4 - 8 + 0 = -4$

2. $f(x,y,z) = (x+1)^2 + (y-2)^2 + (z-1)^2$; $g(x,y,z) = x^2+y^2+z^2-6$

. We solve the system:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \Longrightarrow \begin{cases} 2(x+1) = \lambda \cdot 2x \\ 2(y-2) = \lambda \cdot 2y \\ 2(z-1) = \lambda \cdot 2z \\ x^2+y^2+z^2-6=0 \end{cases} \Longrightarrow \begin{cases} x+1 = \lambda x & \text{—— (1)} \\ y-2 = \lambda y & \text{—— (2)} \\ z-1 = \lambda z & \text{—— (3)} \\ x^2+y^2+z^2 = 6 & \text{—— (4)} \end{cases}$$ 1

. (1) $\Rightarrow \lambda x - x = 1 \Rightarrow x(\lambda - 1) = 1 \Rightarrow x = \dfrac{1}{\lambda-1}$ —— (5) [$\lambda \neq 1$, why?]

(2) $\Rightarrow y - \lambda y = 2 \Rightarrow y(1-\lambda) = 2 \Rightarrow y = \dfrac{2}{1-\lambda}$ —— (6)

(3) $\Rightarrow z - \lambda z = 1 \Rightarrow z(1-\lambda) = 1 \Rightarrow z = \dfrac{1}{1-\lambda}$ —— (7)

. Sub. 5, 6, 7 in (4):

$$\frac{1}{(\lambda-1)^2} + \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} = 6$$

$\Rightarrow 1 + 4 + 1 = 6(\lambda-1)^2$, $(1-\lambda)^2 = (\lambda-1)^2$

$\Rightarrow (\lambda-1)^2 = 1$

$\Rightarrow \lambda - 1 = \pm 1 \Longrightarrow \boxed{\lambda = 0, 2}$ 2

. $\lambda = 0 \xrightarrow{5,6,7} (x,y,z) = (-1, 2, 1)$

$\lambda = 2 \xrightarrow{5,6,7} (x,y,z) = (1, -2, -1)$ 1

. $f(-1, 2, 1) = 0$ ← the min value of f 1

$f(1, -2, -1) = 4 + 16 + 4 = 24$ ← the max value of f

Math201.02, Quiz #3, Term 173

Name: Solutions

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1. [5 points] Find the local max/min values and saddle points of

$$f(x,y) = 2x^2 + y^4 + 4xy - 4.$$

2. [5 points] Find the extreme values of $f(x,y) = \frac{1}{3}x^3 + y^2$ on the circle $x^2 + y^2 = 1$.

Good luck,

Ibrahim Al-Rasasi

[1] $f_x(x,y) = 4x + 4y$; $f_y(x,y) = 4y^3 + 4x$

$\left\{\begin{array}{l} f_x = 0 \\ f_y = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} 4x + 4y = 0 \\ 4y^3 + 4x = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{ll} x + y = 0 & \text{— (1)} \\ y^3 + x = 0 & \text{— (2)} \end{array}\right.$ 1

$(1) \Rightarrow y = -x \overset{(2)}{\Longrightarrow} -x^3 + x = 0 \Rightarrow x(-x^2+1) = 0 \Rightarrow x = 0, -1, 1$

$x = 0 \Rightarrow y = 0$

$x = -1 \Rightarrow y = 1 \quad \Rightarrow \boxed{(x,y) = (0,0),\ (-1,1),\ (1,-1)}$ 1.5

$x = 1 \Rightarrow y = -1$

f_x & f_y exist at all points

Test: $f_{xx}(x,y) = 4$, $f_{yy}(x,y) = 12y^2$; $f_{xy}(x,y) = 4$ 1

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 4 \cdot 12y^2 - 4^2 = 16(3y^2 - 1)$

- $(0,0)$: $D(0,0) = 16(0-1) = -16 < 0 \Rightarrow f$ has a saddle pt at $(0,0)$
- 1.5 $(-1,1)$: $D(-1,1) = 16(3-1) = 32 > 0$ & $f_{xx}(-1,1) = 4 > 0$

  $\Rightarrow f$ has a local min at $(-1,1)$
- $(1,-1)$: $D(1,-1) = 16(3-1) = 32 > 0$ & $f_{xx}(1,-1) = 4 > 0$

  $\Rightarrow f$ has a local min at $(1,-1)$

The local min values of f are $f(-1,1) = 2 + 1 - 4 - 4 = -5$

& $f(1,-1) = 2 + 1 - 4 - 4 = -5$

$\boxed{2}$ $f(x,y) = \frac{1}{3}x^3 + y^2$; $g(x,y) = x^2 + y^2 - 1$

- We solve the system:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Rightarrow \begin{cases} x^2 = \lambda \cdot 2x & \text{— (1)} \\ 2y = \lambda \cdot 2y & \text{— (2)} \\ x^2 + y^2 - 1 = 0 & \text{— (3)} \end{cases}$$ 1

- $(2) \Longrightarrow y = \lambda y \Longrightarrow y - \lambda y = 0 \Longrightarrow y(1-\lambda) = 0 \Longrightarrow \boxed{y = 0 \quad \text{or} \quad \lambda = 1}$ 1

- $\boxed{y=0} \overset{(3)}{\Longrightarrow} x^2 - 1 = 0 \Longrightarrow x = \pm 1 \Longrightarrow \boxed{(x,y) = (-1,0), (1,0)}$ 1

- $\boxed{\lambda = 1} \overset{(1)}{\Longrightarrow} x^2 = 2x \Longrightarrow x(x-2) = 0 \Longrightarrow x = 0, 2$ 1

  - $x = 0 \overset{(3)}{\Longrightarrow} y^2 - 1 = 0 \Longrightarrow y = \pm 1 \Longrightarrow \boxed{(x,y) = (0,-1), (0,1)}$
  - $x = 2 \overset{(3)}{\Longrightarrow} 4 + y^2 - 1 = 0 \Longrightarrow y^2 = -3$, Complex roots. , $y = \pm i\sqrt{3}$ rejected.

- $f(-1,0) = -\frac{1}{3} + 0 = -\frac{1}{3}$

  $f(1,0) = \frac{1}{3} + 0 = \frac{1}{3}$

  $f(0,-1) = 0 + 1 = 1$

  $f(0,1) = 0 + 1 = 1$

- The max. value of f is 1 & it occurs at $(0, \pm 1)$ 1

  The min ______ $-\frac{1}{3}$ ______ $(-1, 0)$