Chapter 3 MQR Markov Chain review

\[ p_{ij} = P(X_{n+1} = j | X_n = i) \]
\[ \sum_{j=1}^{m} p_{ij} = 1 \]

\[ \mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix} \]

\[ \pi_n = (\pi_{1n}, \pi_{2n}, \cdots, \pi_{mn}) \]
\[ \sum_{i=1}^{\pi_n} \pi_{in} = 1 \]

\[ \pi_n \cdot \mathbf{P} = \pi_n \cdot \mathbf{P} \cdots = \pi_n \cdot \mathbf{P}^{r} \quad \text{homogeneous} \]
\[ \pi_n \cdot \mathbf{P}(0) \cdot \mathbf{P}(1) \cdots \cdot \mathbf{P}^{(r-1)} \quad \text{non-homogeneous} \]

\[ rP_{ij}^{(n)} = P(X_{n+r} = j | X_n = i) \]
\[ rP_{ii}^{(n)} = \leq rP_{ij}^{(n)} \] if cannot reenter state i once left i

\[ \lambda_i(j) = \text{force of transition} \]
\[ \lambda_i(s) = \sum_{j=1}^{m} \lambda_{ij}(s) + \sum_{j=i+1}^{m} \lambda_{ji}(s) = \text{force of transitioning out} \]

\[ \frac{d}{dr}(rP_{ij}^{(t)}) = \sum_{k \neq j} \left( rP_{ik}^{(t)} \lambda_{kj}(t+r) - rP_{ij}^{(t)} \lambda_{jk}(t+r) \right) = \sum_{k \neq j} \left( rP_{ik}^{(t)} \lambda_{kj}(t+r) \right) - rP_{ij}^{(t)} \lambda_{j}(t+r) \]

Chapter 12 MQR Multiple Life Functions

The Joint Life and Last Survivor Stuties

\[ T_{xy} = \min(T_x, T_y) \]
\[ f_{xy}(t) = \frac{t}{\mu_{x+y}} f_{x+y}(t) \]
\[ i_p_{xy} = S_{xy}(t) = \exp \left( -\int_0^t \mu_{x+y+r} dr \right) \]

Fundamental Symmetric Relations

Random Variable
\[ T_{xy} + T_{y} = T_x + T_y \]

Survival Function
\[ \delta_{xy} + \delta_{y} = \delta_{x} + \delta_{y} \]

Distribution Function
\[ q_{xy} + q_{y} = q_{x} + q_{y} \]

Density Function
\[ f_{xy}(t) + f_{y}(t) = f_{x}(t) + f_{xy}(t) \]

Defered probability for last survivor:
\[ P(m < K_{xy} < m+n) = m_{n}q_{xy} = m_{n}q_{y} - m_{n}q_{xy} \]

Two Independent Lifetimes
\[ \mu_{x+y+t} = \mu_{x+t} + \mu_{y+t} \]
\[ i_p_{xy} = \Pr(T_x > t) + i_p_{y} \]
\[ \delta_{xy} = \Pr(T_x \leq t \text{ and } T_y \leq t) = i_q_{x} i_q_{y} \]

Fundamental Symmetric Relations (from min(a,b) + max(a,b) = a + b)

\[ E(K_{xy}) = \int_0^\infty k f_{xy}(t)dt = \int_0^\infty k i_p_{xy} dt \]
\[ \delta_{xy} = E(T_{xy}) = \int_0^\infty t \cdot f_{xy}(t)dt = \int_0^\infty t \cdot i_p_{xy} dt \]
\[ \delta_{xy} = E(K_{xy}) = \int_0^\infty k \delta_{xy} dt = \delta_{x} + \delta_{y} - \delta_{xy} \]

\[ E(T_{xy}^2) = \int_0^\infty \frac{t^2}{2} f_{xy}(t)dt = \int_0^\infty \frac{t^2}{2} i_p_{xy} dt \]
\[ E(T_{xy}) = \int_0^\infty \frac{t}{2} f_{xy}(t)dt = \int_0^\infty \frac{t}{2} i_p_{xy} dt \]

\[ E(T_{xy}) = E(T_x) + E(T_y) \]
\[ Var(T_{xy}) = Var(T_x) + Var(T_y) + 2 \cdot \text{Cov}(T_x, T_y) \]

\[ Cov(T_x, T_y) = Cov(T_x, T_y) + [E(T_x) - E(T_{xy})] \times [E(T_y) - E(T_{xy})] \]

\[ Cov(T_{xy}, T_{xy}) = Cov(T_x, T_y) + [E(T_x) - E(T_{xy})] \times [E(T_y) - E(T_{xy})] \]

\[ \delta_{xy} = \int_0^\infty \Pr(T_x > t \text{ and } T_y > u) f_y(u)du \]
\[ \delta_{xy} = \int_0^\infty \Pr(T_x > t \text{ and } T_y > u) f_y(u)du \]

\[ \delta_{xy} = \int_0^\infty \Pr(T_x < t \text{ and } T_y > u) f_y(u)du \]
\[ \delta_{xy} = \int_0^\infty \Pr(T_x < t \text{ and } T_y > u) f_y(u)du \]

Symmetric Relation between Joint and Last Survivor Continuous Insurance
\[ \mathbf{A}_{xy} + \mathbf{A}_{y} = \mathbf{A}_{x} + \mathbf{A}_{xy} \]

similar relations hold for n-year term, pure endowment, and endowment insurance.

Covariance between Joint and Last Survivor Benefits
\[ Cov(u_t^{x}, v_t^{xy}) = Cov(u_t^{xy}, v_t^{x} + (\mathbf{A}_{x} - \mathbf{A}_{xy}) (\mathbf{A}_{y} - \mathbf{A}_{xy}) \]

Similiar relations hold for n-year term, pure endowment, and endowment insurance.

1. Relation between Insurances and Annuities
\[ a_{xy} = \frac{1 - A_{xy}}{d}, \quad a_{xy} = \frac{1 - A_{xy}}{d}, \quad a_{xy} = \frac{1 - A_{xy}}{d} \]

similar relations hold for n-year term insurances and annuities.

2. Fully Discrete Insurances and Annuities
\[ \ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k k p_{xy} \] 
\[ A_{xy} = E \left[ v^{K_{xy}} \right] = \sum_{k=1}^{\infty} v^k \kappa_{xy} \] 
\[ \ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k k p_{xy} \] 
\[ A_{xy} = E \left[ v^{K_{xy}} \right] = \sum_{k=1}^{\infty} v^k \kappa_{xy} = A_x + A_y - A_{xy} \]

3. Reversionary Annuities (payment only when one life fails until the other also fails)

Payment to \( (y) \) when \( (x) \) has failed: 
\[ a_{x|y} = \sum_{k=1}^{\infty} v^k (k p_x - k p_y) = a_y - a_{xy} \]

n-years (at most) pay to \( (x) \) when \( (y) \) has failed: 
\[ a_{y|y} = \sum_{k=1}^{\infty} v^k (k p_x - k p_y) = a_y - a_{xy} \]

Continuous Payment to \( (y) \) when \( (x) \) has failed: 
\[ \ddot{a}_{x|y} = \int_0^\infty v^k (k p_x - p_y) dt = \int_0^\infty v^k (p_y - p_x) dt = \ddot{a}_y - \ddot{a}_{xy} \]

\[ P(a_{y|x}) = \frac{a_{y|x}}{a_{xy}} = \frac{a_y - a_{xy}}{a_{xy}} \]  
\[ tV(a_{y|x}) = \begin{cases} 
A_{y|x} - tP(a_{y|x}) \cdot \ddot{a}_{x+t|y+t} & \text{both survives} \\
0 & \text{if } (x) \text{ survives and } (y) \text{ fails} \\
 & \text{if } (x) \text{ fails and } (y) \text{ survives} 
\end{cases} \]

4. Contingent Insurance

\[ A_{xy} = \int_0^\infty v^t \cdot p_{xy} \mu_{x+t} dt \]  
\[ A_{xy} = \left[ A_{xy} \right] + A_{xy} = A_{xy} \]

5. m-thly payable multiple life benefits

under UDD: 
\[ a_{xy}^{(m)} \approx \alpha(m) a_{xy} - \beta(m) \]
\[ \ddot{a}_{xy}^{(m)} = \alpha(m) \ddot{a}_{xy} - \beta(m) \]
\[ A_{xy}^{(m)} \approx \frac{i}{i(m)} A_{xy} \]

non-UDD: 
\[ a_{xy}^{(m)} \approx \frac{a_{xy} - m - 1}{2m} - \frac{m^2 - 1}{12m^2} (\delta + \mu_{xy}) \]  
\[ \lim_{m \to \infty} a_{xy}^{(m)} = \ddot{a}_{xy} \approx a_{xy} - \frac{1}{2} - \frac{1}{12} (\delta + \mu_{xy}) \]

Woolhouse formula

Prems and Reserves

\[ P_{xy} = \frac{A_{xy}}{a_{xy}} \]
\[ tV_{xy} = A_{x+t|y+t} - P_{xy} \cdot \ddot{a}_{x+t|y+t} \]

\[ P_{xy} = \frac{A_{xy}}{a_{xy}} \]
\[ tV_{xy} = \begin{cases} 
A_{x+t|y+t} - P_{xy} \cdot \ddot{a}_{x+t|y+t} & \text{if } (x) \text{ and } (y) \text{ survives} \\
A_{x+t} - P_{xy} \cdot a_{x+t} & \text{if } (x) \text{ survives and } (y) \text{ fails} \\
A_{x+y+t} - P_{xy} \cdot a_{x+y+t} & \text{if } (x) \text{ fails and } (y) \text{ survives} 
\end{cases} \]

Dependent Life Models - Common Shock Model

\[ \mu_{x+t} = \mu_{x+t}^* + \mu_t^* \]  
\[ \mu_{y+t} = \mu_{y+t}^* + \mu_t^* \]  
\[ \mu_{x+y+t} = \mu_{x+t}^* + \mu_{y+t}^* + \lambda \]  
\[ \mu_{xy+t} = \mu_{x+t}^* + \mu_{y+t}^* + \lambda \]  
\[ \mu_{x+t} = \mu_{x+t}^* + \mu_t^* \]  
\[ \mu_{y+t} = \mu_{y+t}^* + \mu_t^* \]  
\[ \mu_{x+y+t} = \mu_{x+t}^* + \mu_{y+t}^* + \lambda \]  
\[ \mu_{xy+t} = \mu_{x+t}^* + \mu_{y+t}^* + \lambda \]  
\[ \mu_{x+t} = \mu_{x+t}^* + \mu_t^* \]  
\[ \mu_{y+t} = \mu_{y+t}^* + \mu_t^* \]

\[ tV_{xy} = \exp(-t \int_0^t [\mu_{x+t}^* + \mu_{y+t}^* + \lambda]) dt) = t p_{x+t} \cdot t p_{y+t} \cdot e^{-\lambda t} \]

\[ 3. \text{ Discrete Multiple Decrement Models} \]

OBJECTIVES: 1. To understand the concept of a multiple decrement table
2. To understand the force of decrement
3. To construct a multiple decrement model using associated single decrements and to apply various assumptions to calculate rates for discrete jumps.
\[ q^{(r)}_x = q^{(1)}_x + q^{(2)}_x + \ldots + q^{(m)}_x = \sum_{j=1}^{m} q^{(j)}_x \]  
(13.1)

\[ p^{(r)}_x = 1 - q^{(r)}_x \]  
(13.2)

\[ d^{(r)}_x = \sum_{j=1}^{m} d^{(j)}_x = \ell^{(r)}_x \cdot q^{(r)}_x \]  
(13.3 & 13.7b)

\[ n^{(r)}_x = 1 - n^{(r)}_x \]  
(13.7e)

\[ n^{(j)}_x = \sum_{t=0}^{j} d^{(j)}_{x-t} = \ell^{(j)}_x \cdot n^{(j)}_x \]  
(13.4 & 13.7c)

\[ d^{(j)}_x = \ell^{(j)}_x \cdot q^{(j)}_x \]  
(13.7a)

\[ \ell^{(j)}_x = \frac{n^{(j)}_x}{q^{(j)}_x} \]  
(13.7f)

\[ n^{(r)}_x = \sum_{j=1}^{m} n^{(j)}_x = \ell^{(r)}_x \]  
(13.5a & 13.7d)

\[ n^{(r)}_x = \sum_{j=1}^{m} n^{(j)}_x = 1 - n^{(j)}_x \]  
(13.5b)

13.1.2 Random Variable Analysis

The joint probability function of \( K^*_x \) and \( J_x \) is \( \Pr(K^*_x = k \cap J_x = j) = k \cdot q^{(j)}_x \)  
(13.8)

The marginal probability functions are

i. \( \Pr(K^*_x = k) = \sum_{j=1}^{m} \Pr(K^*_x = k \cap J_x = j) = k \cdot q^{(1)}_x + \ldots + q^{(m)}_x = \sum_{j=1}^{m} \frac{d^{(j)}_x}{\ell^{(j)}_x} \)  
(13.9)

ii. \( \Pr(J_x = j) = \sum_{k=1}^{\infty} \Pr(K^*_x = k \cap J_x = j) = \sum_{k=1}^{\infty} \frac{d^{(k)}_x}{\ell^{(k)}_x} \)  
(13.10)

13.2 Theory of Competing Risks

13.3 Continuous Multiple Decrement Models

\[ \mu^{(j)}_{x+t} = -\frac{d}{dt} \ln p^{(j)}_x \]  
(13.12a)

\[ \mu^{(j)}_{x+t} = -\frac{d}{dt} \ln p^{(j)}_x = -\frac{d}{dt} \ln \left[ p^{(1)}_x \cdot p^{(2)}_x \cdot \ldots \cdot p^{(m)}_x \right] \]  
(13.13a)

\[ q^{(j)}_x = \int_0^t s p^{(j)}_x \cdot \mu^{(j)}_{x+s} \cdot ds = 1 - p^{(j)}_x \]  
(13.14)

\[ f^{(j)}_x(t) = t^{(j)}_x \cdot \mu^{(j)}_{x+t} \]  
(13.14)

\[ F^{(j)}_x(t) = \Pr \left[ r^{(j)}_x \leq t \right] = \int_0^t f^{(j)}_x(s) \cdot ds = \int_0^t s p^{(j)}_x \cdot \mu^{(j)}_{x+s} \cdot ds \]  
(13.15)

\[ \mu^{(r)}_{x+t} = -\frac{d}{dt} \ln p^{(r)}_x = -\frac{d}{dt} \ln \left[ q^{(1)}_x \cdot q^{(2)}_x \cdot \ldots \cdot q^{(m)}_x \right] \]  
(13.17)

Fundamental Relation Between Primed and Unprimed Rates: \( t p^{(r)}_x = \exp \left( -\sum_{j=1}^{m} \int_0^t s p^{(j)}_x \cdot \mu^{(j)}_{x+s} \cdot ds \right) = \prod_{j=1}^{m} t p^{(j)}_x \)  
(13.16)

\[ t q^{(j)}_x = \int_0^t r t(s, j) \cdot ds = \int_0^t s p^{(r)}_x \cdot \mu^{(j)}_{x+s} \cdot ds \]  
(13.18)

\[ t q^{(j)}_x = \int_0^t d t^{(j)}_x = \int_0^t s p^{(r)}_x \cdot \mu^{(j)}_{x+s} \cdot ds \]  
(13.19)

Joint Distribution of \( T_x \) and \( J_x \) \( \Pr(t < T_x \leq t + dt \text{ and } J_x = j) \approx t p^{(r)}_x \cdot \mu^{(j)}_{x+t} \cdot dt \cdot t q^{(j)}_x = \int_0^t s p^{(r)}_x \cdot \mu^{(j)}_{x+s} \cdot ds \)  
(13.20)

13.4.1 Uniform Distribution of Decrement in the Multiple Decrement Context

\[ t \cdot q^{(r)}_x = q^{(r)}_x \]  
(13.21)

\[ t \cdot q^{(j)}_x = q^{(j)}_x \cdot \mu^{(r)}_{x+t} \]  
(13.22)

\[ t \cdot q^{(j)}_x = q^{(j)}_x \cdot \mu^{(r)}_{x+t} = 1 - t \cdot q^{(r)}_x \]  
(13.23)

\[ t \cdot q^{(r)}_x = \frac{q^{(r)}_x}{1 - t \cdot q^{(r)}_x} \]  
(13.24)

\[ t \cdot q^{(j)}_x = \frac{q^{(j)}_x}{1 - t \cdot q^{(r)}_x} \]  
(13.25)

\[ t p^{(r)}_x = \exp \left[ \frac{q^{(j)}_x}{q^{(r)}_x} \cdot \ln \left( 1 - t \cdot q^{(r)}_x \right) \right] = \left( 1 - t \cdot q^{(r)}_x \right)^{n^{(j)}_x / q^{(r)}_x} \]  
(13.26)

13.4.2 Uniform Distribution in the Associated Single-Decrement Tables

\[ t q^{(j)}_x = t \cdot q^{(j)}_x \]  
(13.27)

\[ t p^{(j)}_x \cdot \mu^{(j)}_{x+t} = q^{(j)}_x \]  
(13.28)

<table>
<thead>
<tr>
<th>Decrement</th>
<th>( t q^{(1)}_x = 0 )</th>
<th>( t q^{(2)}_x = 0 )</th>
<th>( t q^{(3)}_x = 0 )</th>
<th>( t q^{(4)}_x = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double</td>
<td>( t q^{(1)}_x = 0 )</td>
<td>( t q^{(2)}_x = 0 )</td>
<td>( t q^{(3)}_x = 0 )</td>
<td>( t q^{(4)}_x = 0 )</td>
</tr>
<tr>
<td>Triple</td>
<td>( t q^{(1)}_x = 0 )</td>
<td>( t q^{(2)}_x = 0 )</td>
<td>( t q^{(3)}_x = 0 )</td>
<td>( t q^{(4)}_x = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Decrement</th>
<th>( t q^{(1)}_x = 0 )</th>
<th>( t q^{(2)}_x = 0 )</th>
<th>( t q^{(3)}_x = 0 )</th>
<th>( t q^{(4)}_x = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double</td>
<td>( t q^{(1)}_x = 0 )</td>
<td>( t q^{(2)}_x = 0 )</td>
<td>( t q^{(3)}_x = 0 )</td>
<td>( t q^{(4)}_x = 0 )</td>
</tr>
<tr>
<td>Triple</td>
<td>( t q^{(1)}_x = 0 )</td>
<td>( t q^{(2)}_x = 0 )</td>
<td>( t q^{(3)}_x = 0 )</td>
<td>( t q^{(4)}_x = 0 )</td>
</tr>
</tbody>
</table>

Miscellaneous Results (From ACTEX MLC manual)
1. Assumptions on the single decrement table.

Back out the Unprimed Rates from Primed Rates

\[ q_x^{(i)} = \int_0^s r p_x^{(r)} s p_x^{(i)} \, dt = s p_x^{(r)} \left[ \prod_{j=1, j \neq i}^{m} \int_0^{t} r p_x^{(j)} \, dt \right] r p_x^{(i)} \, \mu_{x+t} \, dt \]

2. Constant Force Assumption for Multiple Decrements

(i) For any \( t \in [0, 1] \) and integer-valued \( x \),

\[ q_x^{(r)} = \left[ p_x^{(r)} \right]^t \] (survival probability for fractional ages)

(ii) Ratio Property : \( \frac{t q_x^{(i)}}{r q_x^{(r)}} = \frac{\mu_{x+s}}{\mu_x} \) for any \( s \in [0, 1] \)

(iii) Partition Property : \( q_x^{(i)} = \left[ p_x^{(r)} \right]^{q_x^{(r)}} \) (To get primed rates from unprimed rates from (i))

3. Uniform Distribution of Death (UDD) for Multiple Decrement (MUDD) Table

For any \( t \in [0, 1] \) and integer-valued \( x \),

(i) \( q_x^{(r)} \mu_x^{(i)} = q_x^{(i)} \) or equivalently \( \mu_x^{(i)} = \frac{q_x^{(i)}}{1 - q_x^{(r)}} \) for \( t \neq 1 \)

(ii) Ratio Property : \( \frac{t q_x^{(i)}}{r q_x^{(r)}} = \frac{\mu_{x+s}}{\mu_x} \) for any \( s \in [0, 1] \)

(iii) Partition Property : \( q_x^{(i)} = \left[ p_x^{(r)} \right]^{q_x^{(r)}} \) (To get primed rates from unprimed rates \( t q_x^{(i)} \) and \( r p_x^{(r)} \))

Discrete jumps: Handling Both Discrete and Continuous Decrement

1)  \( s q_x^{(i)} = \int_0^s \left[ \prod_{j=1, j \neq i}^{m} \int_0^{t} r p_x^{(j)} \, dt \right] r p_x^{(i)} \mu_{x+t} \, dt \) holds when decrement \( i \) is continuous.

2)  \( s q_x^{(i)} = \sum_{s \leq t} \left[ \prod_{j=1, j \neq i}^{m} \int_0^{t} r p_x^{(j)} \, dt \right] \Delta (t, q_x^{(i)}) \) holds when decrement \( i \) is discrete.

Here \( t_k \) are the jump times and \( \Delta (t, q_x^{(i)}) \) is the jump size at time \( t_k \).

Chapter 14 MQR Multiple Decrement Models: (Applications)

14.1 Actuarial Present Value

\[ A_x = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k) \] (14.1) \[ \quad A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k \cap J_x = j) \] (14.2)

If the time and cause of decrement are independent,

\[ A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot \Pr(K_x^* = k) \cdot \Pr(J_x = j) \] (14.3a) or \[ A_x^{(j)} = \sum_{k=1}^{\infty} v^k \cdot k \cdot p_x^{(r)} \cdot q_x^{(i)} \] (14.3b)

For benefit paid at the instant of failure \( \overline{A}_x^{(j)} = \int_0^{\infty} v^t \cdot r p_x^{(r)} \cdot \mu_{x+t} \, dt \) (14.4)

14.2 Asset Shares \( [a AS + G(1 - r_t) - c_k(1 + i)] + b_k(1) \cdot q_x^{(1)} + b_k(2) \cdot q_x^{(2)} + 1 AS \cdot p_x^{(r)} \),

so \( 1 AS = \left[ a AS + G(1 - r_t) - c_k(1 + i) \right] + b_k(1) \cdot q_x^{(1)} + b_k(2) \cdot q_x^{(2)} \) (14.5a)

In general, \( [k - 1 AS + G(1 - r_t - c_k(1 + i)] = b_k(1) \cdot q_x^{(1)} + b_k(2) \cdot q_x^{(2)} + k AS \cdot p_x^{(r)} \),

so \( k AS = \left[ k - 1 AS + G(1 - r_t - c_k(1 + i)] = b_k(1) \cdot q_x^{(1)} + b_k(2) \cdot q_x^{(2)} + k AS \cdot p_x^{(r)} \right] \) (14.6b)

14.3 Non-Forfeiture Options

14.3.1 Cash Value \( tCV_x \)

14.3.2 Reduced Paid-up Insurance

\[ RPU = \frac{tCV_x}{A_x^{(t)}} \] (14.8) \[ tW_x = \frac{tCV_x}{A_{x+t}} \] (14.9)

14.3.3 Extended Term Insurance \( tCV_x = A_x^{(t)} \) (14.10) \[ tCV_{x:n} = A_x^{(t)} + PE \cdot n \cdot E_{x+t} \) (14.11)

14.4 Multi State Model Representation

14.4.2 The Total and Permanent Disability Model

\[ h \overline{A}_x^{(j)} = \int_0^{\infty} v^r \cdot d \overline{A}_x^{(r)} \cdot k \cdot p_x^{(r)} \cdot \mu_{x+t} \, dt \] (14.12a) \[ h \overline{A}_x^{(j)} = \int_0^{\infty} v^r \cdot d \overline{A}_x^{(r)} \cdot k \cdot p_x^{(r)} \cdot \mu_{x+t} \, dt \] (14.12b)

\[ h \overline{A}_x^{(j)} = \int_0^{\infty} v^r \cdot d \overline{A}_x^{(r)} \cdot k \cdot p_x^{(r)} \cdot \mu_{x+t} \, dt \] (14.12c)

14.4.3 The Total and Permanent Disability Model

\[ h \overline{A}_x^{(j)} = \int_0^{\infty} v^r \cdot d \overline{A}_x^{(r)} \cdot k \cdot p_x^{(r)} \cdot \mu_{x+t} \, dt \] (14.13a) \[ h \overline{A}_x^{(j)} = \int_0^{\infty} v^r \cdot d \overline{A}_x^{(r)} \cdot k \cdot p_x^{(r)} \cdot \mu_{x+t} \, dt \] (14.13b)
14.4.5 Thiele’s Differential Equation

Gain from factor whose gain is calculated first is

\[ G^{(1)} = P(1) - P(0) \]

(14.40a)

Gain from factor whose gain is calculated second is

\[ G^{(2)} = P(2) - P(1) \]

(14.40b)

Gain from factor whose gain is calculated third is

\[ G^{(3)} = P(3) - P(2) \]

(14.40c)

Gain from factor whose gain is calculated fourth is

\[ G^{(4)} = P(4) - P(3) \]

(14.40d)

Gain from factor whose gain is calculated kth is

\[ G^{(k)} = P(k) - P(k-1) \]

Total gain \( G^T = G^{(1)} + G^{(2)} + G^{(3)} + G^{(4)} = P(4) - P(0) \)

When death occurs throughout year but withdrawal only at end of year, the anticipated profit expression is

\[ P(0) = [V + G(1 - e_{i+1})(1 + i_{i+1})] - \left[ \left( b_{i+1}^{(1)} + s_{i+1}^{(1)} \right) \cdot q_{x+i+1}^{(1)} + \left( b_{i+1}^{(2)} + s_{i+1}^{(2)} \right) \right] \cdot q_{x+i+1}^{(2)} + p_{x+i+1}^{(r)} \cdot t_{i+1} V \]

(14.41)

14.5 Defined Benefit (DB) Pension Plans

14.5.1 Normal Retirement (NR) Benefits
**Projected Annual Benefit**

\[ PAB_x = 0.01 p \cdot YOS_x \cdot FAS_x \]  

(14.29)

**Final Annual Salary**

\[ FAS_x = \frac{1}{3} \left( \sum_{k=0}^{z-1} S_k \cdot CAS_x \right) \]  

(14.30)

**Projected Aggregate Salary**

\[ PAS_x = \frac{1}{S_x} \sum_{k=x}^{\infty} S_k \cdot CAS_x \]  

(14.31)

**Projected Annual Retirement Benefit**

\[ PAB_x = 0.01 p \cdot PAS_x \]  

(14.32)

\[ APV \] of the projected benefit, at age \( x \)

\[ APV_x^{NR} = PAB_x \cdot v^{z-x} \cdot z \cdot p_x^{(r)} \cdot r^{(12)} \]  

(14.33)

14.5.2 Early Retirement (ER) Benefits

\[ APV_{ER}^{W5} = \sum_{y=60}^{64} PAB_{y+1/2} \left( 1 - 0.05 \left( 65 - y - \frac{1}{2} \right) \right) \cdot v^{y+1/2-35} \cdot y \cdot 35 \cdot p_{35}^{(r)} \cdot q_y^{(w)} \cdot w \cdot 65-y-1/2 \cdot p_{y+1/2}^{(r)} \cdot r^{(12)} \]  

(14.34)

14.5.3 Withdrawal and Other Benefits

Assuming a 5-year vesting rule and assuming employees take their withdrawal benefit at NRA, the APV at age 35 is

\[ APV_{35}^W = \sum_{y=30+5}^{50} PAB_{y+1/2} \cdot v^{30} \cdot y \cdot 35 \cdot p_{35}^{(r)} \cdot q_y^{(w)} \cdot w \cdot 65-y-1/2 \cdot p_{y+1/2}^{(r)} \cdot r^{(12)} \]  

(14.35)

14.5.4 Funding and Reserving

**Normal Cost (Early Age)**

\[ NC_x^{EAN} = \frac{APV_x^{T}}{u^{(r)}(z-x)} \]  

(14.36)

\[ tV_x = APV_x^{T} - NC_x^{EAN} \cdot \tilde{a}_x^{(r)} \]  

(14.37a) or retrospectively as

\[ tV_x = NC_x^{EAN} \cdot \tilde{s}_x^{(r)} \]  

(14.37b)

**APV of the benefit accrued between ages \( x \) and \( x+1 \):**

\[ APV_x^{NR} = \left( AB_x^{x+1} - AB_x \right) \cdot v^{z-x} \cdot z \cdot p_x^{(r)} \cdot r^{(12)} \]  

(14.38)

**ACTEX MLC Chapter 9 Study Manual Vol II**  
Multiple Decrement Models: Applications

Thiele’s Differential Equation under Multiple Decrement

\[ \frac{dV^g}{dt} = G_t - c_t + \left( \delta + \mu_t^{(r)} \right) V^g - \sum_{j=1}^{n} \left( b_{t}^{(j)} + E_t^{(j)} \right) \mu_t^{(j)} \]  

Recursive Relation for Expected Asset Shares

\[ h \cdot AS + G \cdot h \cdot (1 - c_h) - e_h \cdot (1 + i) = p_x^{(r)} \cdot h \cdot AS + q_h^{(1)} \cdot h \cdot CV + q_h^{(2)} \cdot b_{h+1} \]  

**ACTEX MLC Chapter 16 Pension Mathematics**

16.1 Salary Scale Function

\[ \frac{s_y}{s_x} = \frac{\text{Salary received in year of age } y \text{ to } y+1}{\text{Salary received in year of age } x \text{ to } x+1} = \int_{y}^{y+1} \frac{s_y}{s_x} du \]  

\[ \int_{x}^{x+1} \frac{s_y}{s_x} du \]  

Salary received in year of age \( y \) to \( y+1 \) = (Salary received in year of age \( x \) to \( x+1 \)) \( \frac{s_y}{s_x} \approx \) (Salary received at age \( x \)) \( \frac{s_y}{s_x-0.5} \)

**Approximations**, idea: Salary rate at age \( x \approx s_x-0.5 \)

Salary received in year of age \( y \) to \( y+1 \) = (Salary received at age \( x \)) \( \frac{s_y}{s_x} \approx \) (Salary received at age \( x \)) \( \frac{s_y}{s_x-0.5} \)

16.2 Salary Scale Function

**Defined Contribution (DC) Plan**

future retirement benefit is subject to investment earnings during accumulation phase

**Defined Benefit (DB) Plan**

future retirement benefit is defined

16.3 Setting the DC Contribution Rate

Finding the appropriate contribution rate for a DC plan to meet a target replacement ratio \( R \), the following are needed

1. target replacement ratio \( R \) and target retirement age \( z \)
2. assumptions on (1) investment rate of return, (2) interest rates at retirement, (3) a salary scale and (4) post-retirement mortality model
3. the form the benefits should take

**Step 1** Project future salaries. Use the salary scale.

2) At retirement date, calculate accumulated value of all contributions. Don’t use mortality and withdrawal as DC fund is like a bank account. If a person dies, the accumulated fund is paid out to beneficiaries. If a person withdraws, the accumulated fund is paid to the employee.

3) Multiply final salary (in step 2) by the replacement ratio to get the annual pension benefits. Then calculate the actuarial present value of all pension benefit at retirement.

4) Solve for contribution rate by equating values in Step 2 and Step 3.

Handy formula for DC plan:

\[ \sum_{k=1}^{n} x^k = \frac{x^{n+1} - x}{x - 1} \]
16.4 DB Plan and Service Table

<table>
<thead>
<tr>
<th>Pension Plan</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Salary Plan</td>
<td>( R(x, h, t) = \alpha (AS_{x+h+t-1}) = \alpha \times AS_{x+h-1} \frac{s_{x+h+t-1}}{s_{x+h-1}} )</td>
</tr>
<tr>
<td>Final m-year average salary plan</td>
<td>( R(x, h, t) = \alpha (h + t) \frac{AS_{x+h+t-1} + AS_{x+h+t-2} + \cdots + AS_{x+h+t-m}}{m} )</td>
</tr>
<tr>
<td>Multidecrement model with irreversible states: 0 (Active employee), Projected Unit Credit. Uses the average of Pension Plan 16.4 DB Plan and Service Table</td>
<td></td>
</tr>
<tr>
<td>or Final Average Pay (FAP) plan</td>
<td>( R(x, h, t) = \alpha (h + t) \frac{AS_{x+h+t-1} + AS_{x+h+t-2} + \cdots + AS_{x+h+t-m}}{m} )</td>
</tr>
<tr>
<td>Career Average Earnings (CAE) plan</td>
<td>( R(x, h, t) = \alpha (h + t) \frac{TPE(x, h, t)}{h + t} = \alpha (h + t) CAE(x, h, t) = \alpha TPE(x, h, t) )</td>
</tr>
<tr>
<td>Accumulation phase</td>
<td>( R(x, h, t) = \alpha (AS_{x+h+t-1} + AS_{x+h+t-2} + \cdots + AS_{x+h+t-1}) )</td>
</tr>
<tr>
<td>after transition from state 0</td>
<td>( R(x, h, t) = \alpha (AS_{x+h+t-1} + AS_{x+h+t-2} + \cdots + AS_{x+h+t-1}) )</td>
</tr>
<tr>
<td>3 Single Decrement models</td>
<td>( R(x, h, t) = \alpha (AS_{x+h+t-1} + AS_{x+h+t-2} + \cdots + AS_{x+h+t-1}) )</td>
</tr>
<tr>
<td>Exact APV</td>
<td>( \int_{n_r-x-h}^{\infty} i^t \cdot \varphi_{x+h}^{(r)}(x) \varphi_{t}^{(r)}(t) \varphi_{x+h+t}^{(r)}(x) \frac{R(x, h, t) \cdot r_{x+h+t}^{(12)}}{r_{x+h+t+1/2}} \cdot r_{x+h+t+1/2}^{(12)} \cdot n_{r-x-h} )</td>
</tr>
<tr>
<td>Approximate APV mult decr table</td>
<td>( \sum_{t=n_r-x-h}^{\infty} i^{t+1/2} \cdot \varphi_{x+h}^{(r)}(x) \varphi_{t}^{(r)}(t) \varphi_{x+h+t}^{(r)}(x) \frac{R(x, h, t) \cdot r_{x+h+t}^{(12)}}{r_{x+h+t+1/2}} \cdot r_{x+h+t+1/2}^{(12)} \cdot n_{r-x-h} )</td>
</tr>
</tbody>
</table>

Service Table: Under total probability law, for any integer age \( x > x_0 \), \( l_x = l_{x-1} - w_{x-1} - i_{x-1} - r_{x-1} - d_{x-1} \). (10.2)

For \( x_0 + k (k = 0, 1, \cdots) \), \( w_{x_0+k} = l_{x_0} k^0_0 p_0^0 \frac{p_0^0}{p_0^1} \), \( i_{x_0+k} = l_{x_0} k^0_1 p_0^1 p_0^2 \), \( r_{x_0+k} = l_{x_0} k^0_2 p_0^3 p_0^4 \), \( d_{x_0+k} = l_{x_0} k^0_3 p_0^4 \), \( l_{x_0+k} = l_{x_0} k^0_0 \). 

Pension Plan Replacement Ratio, \( R = \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}} \)

16.5 Funding of DB Plans

| PUC | Projected Unit Credit. Uses the average of final \( m \) salaries |
| TUC | Traditional Unit Credit. Use the most recent \( m \) salaries |

\( i^V + C_t = EPV \) of benefits of mid-year exits + \( v p_x^{00} \cdot t+1 V \) where \( C_t = \) normal cost

\( C_t = EPV \) of benefits of mid-year exits + \( v p_x^{00} \cdot t+1 V - i^V \)

Chapter 15 MQR Models with Variable Interest Rates

15.3 Forward Interest Rates \( 1000 = 1000c(1 + z_{0.5})^{-1} + 1000c \left( 1 + \frac{z_{1.0}}{2} \right)^{-2} + \cdots + 1000(1+c) \left( 1 + \frac{2k}{2} \right)^{-2k} \)

(15.1)

\( 1 + y_k)^5 = (1 + y_k)^1 \cdot (1 + f_{1,4})^4. \)

(15.2) \( 1 + y_k) \cdot (1 + f_{k,5-k}) = (1 + y_k)^{k+5-k} = (1 + y_k)^5. \)

15.4 Forward Interest Rates \( 1000c(1 + z_{0.5})^{-1} + 1000c \left( 1 + \frac{z_{1.0}}{2} \right)^{-2} + \cdots + 1000(1+c) \left( 1 + \frac{2k}{2} \right)^{-2k} \)

Chapter 12 ACTEX MLC Study Manual Vol II Interest Rate Risk

Spot interest rate \( v(t) = (1 + y_t)^{-t} \)

Forward interest rate \( (1 + f_{t,k})^k = \frac{(1 + y_{t+k})^{t+k}}{(1 + y_t)^t} = \frac{v(t)}{v(t + k)} \)

Chapter 13 Profit Testing

| Profit vector | \( \mathbf{Pr} = (Pr_0, Pr_1, Pr_2, Pr_3, \ldots)' \) |
| Profit signature | \( \Pi = (\Pi_0, \Pi_1, \Pi_2, \Pi_3, \ldots)' = (Pr_0, Pr_1, p_x, Pr_2, 2p_x, Pr_3, \ldots)' \) |

Expected profit that emerges in \( (h + 1)^{th} \) year

\( \mathbf{Pr}_{h+1} = N[(h V + G_0(1 - c_h) - e_h)(1 + h_{h+1} - (b_{h+1} + E_{h+1})q_{x+h} + p_{2x+h} h_{h+1} V)] \)

\( = N[(G_h(1 - c_h) - e_h)(1 + h_{h+1} - (b_{h+1} + E_{h+1})q_{x+h} + (1 + i_{h+1})h V - p_{x+h, h+1} V)] \)

(1)
Extension to Multiple State Models (assuming $N = 1$)

$$ Pr_{t}^{(j)} = [h V^{(j)} + G_{h}^{(j)} (1 - c_{h}^{(j)}) - e_{h}^{(j)} (1 + i_{h+1}) - \sum_{k=0}^{n} (b_{h+1}^{(jk)} + E_{h+1}^{(jk)} h_{+1} V^{(k)})]^{k}_{p_{x+h}^{k}} $$

$$ \Pi = (\Pi_{0}, \Pi_{1}, \Pi_{2}, \ldots)' \quad \Pi_{0} = Pr_{0}^{(0)} \quad \Pi_{1} = \sum_{k=0}^{n} 0p_{k}^{0k} Pr_{t}^{(k)} = 0p_{0}^{00} Pr_{1}^{(0)} = Pr_{1}^{(0)} \quad \Pi_{t} = \sum_{k=0}^{n} t-1p_{z}^{0k} Pr_{t}^{(k)} $$

Traditional Insurance Policies with Withdrawal (assuming $N = 1$)

$$ Pr_{h+1}^{(0)} = [h V^{(0)} + G_{h}(1 - c_{h}) - e_{h} (1 + i_{h+1}) - [h_{+1} V^{(0)}]^{P}_{x+h} + (b_{h+1} + E_{h+1}) q_{x+h}^{(1)}] h_{+1} CV_{q_{x+h}^{(2)}} \geq 0 $$

$$ Pr_{h+1}^{(1)} = 0, Pr_{h+1}^{(2)} = 0, h \geq 1 \quad Pr_{0}^{(0)} = - acquisition costs = -pre-contract expenses - oV $$

Extension to Policies with Continuous Benefit (assuming $N = 1$): A crude approximation

$$ Pr_{h+1} = [h V^{0} + G_{h}(1 - c_{h}) - e_{h} (1 + i_{h+1}) - (1 + i_{h+1})^{1/2} (b_{h+0.5} + E_{h+0.5}) q_{x+h}^{(2)} + p_{x+h} h_{+1} V^{(0)}] $$

13.2 Profit Measures

1. Net present value (NPV)

$$ NPV = \sum_{k=0}^{n} \Pi_{k} \left( 1 + r \right)^{k} $$

If the $NPV > 0$, then the investment = profitable

2. Internal rate of return (IRR)

The internal rate of return is the zero of the equation

$$ NPV(r) = \sum_{k=0}^{n} \frac{C_{k}}{(1 + r)^{k}} = 0 $$

3. Discounted payback period (DPP)

Given the hurdle rate $r$, the discounted payback period (also known as the break-even period) is the smallest value of $m$ such that the partial net present value $NPV(r; m) = \sum_{k=0}^{m} \frac{C_{k}}{(1 + r)^{k}} \geq 0$.

DPP is the time until the investment starts to make a profit.

4. Profit margin

$$ \frac{NPV}{acquisition costs} = \frac{1}{\Pi_{0}} \sum_{k=0}^{n} \frac{\Pi_{k}}{(1 + r)^{k}} $$

13.3 Using Profit Test to Compute Premiums and Reserves

Zeroization of profit vector

<table>
<thead>
<tr>
<th>Profit Vector without allowance for reserves</th>
<th>$Pr = (Pr_{0}, Pr_{1}, Pr_{2}, Pr_{3}, \ldots)'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>term policy for $t &lt; n - 1$</td>
<td>$iV^{Z} = \max \left( \frac{p_{x+t} \cdot t+1 V^{Z} - Pr_{t+1}^{(0)}}{1 + i_{t}} \right)$</td>
</tr>
<tr>
<td>endowment policy</td>
<td>same as term policy for $t &lt; n - 1$</td>
</tr>
<tr>
<td></td>
<td>at $t = n$, $n-1 V^{Z} = \max \left( \frac{- Pr_{n}^{(0)}}{1 + i_{n-1}} \right)$</td>
</tr>
<tr>
<td>Profit after zeroization</td>
<td>$Pr_{h+1}^{(0)} = Pr_{h+1}^{(0)} + (1 + i_{h+1}) h_{+1} V^{Z}$</td>
</tr>
</tbody>
</table>

Note: If $i_{+1} V^{Z} = 0$ and $Pr_{t+1} > 0$, $iV^{Z} = \max \left( -ae, 0 \right) = 0$. 

8
Chapter 16 MQR Universal Life Insurance (ULI)

16.1 Definition and Basic Mechanics.

<table>
<thead>
<tr>
<th>Contract Type</th>
<th>Cost of Insurance ((COI_t)) for ULI</th>
<th>for ULI with Corridor Factor ((CF))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(\frac{q_{x+t-1}}{1+i^q} (B - (AV_{t-1} + G_t(1 - r_t) - e_t) (1 + i^c)))</td>
<td>(COI_t^{CF} = \frac{NAR_t \cdot q_{x+t-1}}{1+i^q})</td>
</tr>
<tr>
<td>B</td>
<td>(B \cdot \frac{q_{x+t-1}}{1+i^q})</td>
<td>(\frac{q_{x+t-1}}{1+i^q} (AV_{t-1} + G_t(1 - r_t) - e_t) (1 + i^c) (f - 1))</td>
</tr>
</tbody>
</table>

Death Benefit, Accumulated Value \((AV_t)\) and Net Amount at Risk \((NAR_t)\)

<table>
<thead>
<tr>
<th>ULI Contract Type</th>
<th>Death Benefit</th>
<th>(AV_t) for ULI</th>
<th>(NAR_t) for ULI</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(B)</td>
<td>((AV_{t-1} + G_t(1 - r_t) - e_t - B\frac{q_{x+t-1}}{1+i^q}) (1 + i^c))</td>
<td>(B - AV_t)</td>
</tr>
<tr>
<td>B</td>
<td>(B + AV_t)</td>
<td>((AV_{t-1} + G_t(1 - r_t) - e_t - COI_t) (1 + i^c))</td>
<td>(B + AV_t - AV_t = B)</td>
</tr>
<tr>
<td>A with Corridor factor</td>
<td>(DB_t = \max(B, f \cdot AV_t))</td>
<td>((1 + i^c) \times [AV_{t-1} + G_t(1 - r_t) - e_t - \max(COI_t^{CF}, COI_t^{lo} CF)])</td>
<td>(DB_t - AV_t = \max[B_t - AV_t, (f - 1)AV_t])</td>
</tr>
<tr>
<td>B with Corridor factor</td>
<td>(\max(B + AV_t, f \cdot AV_t))</td>
<td>same formula as above</td>
<td>(DB_t - AV_t = \max[B_t - AV_t, (f - 1)AV_t])</td>
</tr>
</tbody>
</table>

\(i^q\) interest rate in the calculation of premium for death benefit
\(i^c\) interest rate in the calculation of saving part of ULI
\(f\) corridor factor to use for the minimum death benefit or the minimum insurance amount.

**ULI Universal Life Insurance.** Policy is marked by (a) extensive policyholder choice,
(b) policyholder participation in interest rate risk, and (c) secondary guarantee features of coverage

**VUL Variable Universal Life insurance.** Separate investment accounts for net contributions.

**EIUL Equity-Indexed Universal Life insurance.** Interest/investment is credited to contract at rate that depends on some published stock index such as SP500, DJIA, or EAFE index

**SC** Surrender Charge at time \(t\).

**M&E** Mortality and Expenses Charge at time \(t\).

**NAR_t** Net Amount at Risk at time \(t\).

**AV_t** Account Value at time \(t\).

**CV_t** Cash Value at time \(t\). \((CV_t = AV_t - SC_t)\)

**NAIC** National Association of Insurance Commissioners

**PG** Policy Guarantees (Guarantees given as part of an insurance policy).

**GMP** Guaranteed Maturity Premium. Leveled gross premium sufficient to endow the policy at its maturity date based on the policy guarantees of premium loads, interest rates, and expense and mortality charges.

**GMF** Guaranteed Maturity Funds. Calculated based on the roll forward of the GMP and the policy guarantees.

XXX16.2 Indexed Universal Life Insurance,

a) Point-to-point method: \(i_p = \frac{\text{Final Index Closing Value}}{\text{Initial Index Closing Value}} - 1,\) (16.1)

b) Monthly average method: \(i_{MA} = \frac{1}{12} \sum \frac{\text{Monthly Index Closing Values}}{\text{Initial Index Closing Value}} - 1.\) (16.2)

XXX16.3 Pricing Considerations

*Mortality rate, Lapse rate, Expenses, Investment Income.*

Double decrement model: \(p_x^{(r)} = 1 - q_x^{(r)} = 1 - q_x^{(d)} - q_x^{(u)}\). (16.3)

Withdrawal at end of year only: \(p_x^{(r)} = \left(1 - q_x^{(d)}\right) \left(1 - q_x^{(u)}\right)\). (16.4)


XXX16.4 Reserving Considerations
XXX16.4.1 Basic Universal Life (UL)

Process for 1983 NAIC regulation to define a minimum reserving standard for UL products.

a) At policy issue,

1. a guaranteed maturity premium (GMP) is calculated as the level gross premium sufficient to endow the policy at its maturity date. The GMP is based on the policy guarantees of premium loads, interest rates, and expense and mortality charges.

   \[ GMP_0 = \text{policy guarantees of } f(\text{premium loads, } i, M&E) \]

2. a sequence of guaranteed maturity funds (GMF) is calculated based on the roll forward of the GMP and the policy guarantees

   a sequence \[ GMF = \text{roll forward of } f(\text{GMP, policy guarantees}) \]

b) At the valuation date, \( t \),

3. actual \( AV_t \) determined by the account value roll forward process.

   ULI with variable failure benefit \( B + AV_t \) (11.25c): \( AV_t = [AV_{t-1} + G_t(1-r_t) - e_t - v_t q_{x+t-1} B](1+i_t) \)

   ULI with fixed failure benefit (11.26b): \( AV_t = [AV_{t-1} + G_t(1-r_t) - e_t](1+i_t) - q_{x+t-1}(B - AV_t) \)

   (11.27) \[ AV_t = \frac{[AV_{t-1} + G_t(1-r_t) - e_t]}{p_{x+t-1}}(1+i_t) - q_{x+t-1} B \]

4. the ratio of the actual account value to the GMF is calculated as \( r_t = \frac{AV_t}{GMF_t}, \quad r_t \leq 1 \)  \(  \quad (16.5) \)

5. \( \max(AV_t, GMF_t) \) is projected forward based on the GMP and the policy guarantees. This produces a sequence of GDB and GMB.

6. \( PVFB_t \) and \( PVFP_t \) are calculated using valuation assumptions. Then the pre-floor CRVM reserve is defined as

   \[ t^{\text{V pre-floor CRVM}} = r_t(PVFB_t - PVFP_t) \quad (16.6) \quad \text{with } r_t \text{ as defined above.} \]

7. \( t^{\text{V floor CRVM}} = \max(\frac{1}{2}\text{-month term reserve based on minimum valuation mortality and interest, } CSV_t) \).

8. \( t^{\text{V final CRVM}} = \max(t^{\text{V pre-floor CRVM}}, t^{\text{V floor CRVM}}) \).

The regulation also defines alternative minimum reserves (AMR).

1. The valuation net premium is calculated at policy issue (\( t = 0 \)) based on the GMP and the policy guarantees.

2. If the GMP < the valuation net premium (VNP), the reserve held \( \max(a, b) \)

   where \( a \) =the reserve calculated using the actual method and assumptions of the policy + VNP,

   \( b \) = the reserve calculated using the actual method but with minimum valuation assumptions + GMP.

XXX16.4.1 Indexed Universal Life (eIUL)

NAIC Actuarial Guideline 36 (AG 36) specifies the valuation standards for IUL contracts. 3 computational methods:

1) The implied guaranteed rate (IGR) method: which requires insurers to satisfy the hedged-as-required criteria. These criteria set forth a strenuous constraint requiring exact, or nearly exact, hedging, as well as an indexed
interest-crediting term of not more than one year.  

2) The CRVM with updated market value (CRVM/UMV) method:  
   must be used if the contract has an indexed interest-crediting term of more than one year, or if the renewal participation rate guarantee gives an implied guaranteed rate greater than the maximum valuation rate. This method can be volatile when market conditions change.  

(3) The CRVM with updated average market value (CRVM/UAMV) method:  
   is a hybrid of the other two, designed for an insurer who qualifies for the first method above but does not wish to satisfy the hedged-as-required criteria.  

The CRVM/UMV method has calculation steps as follows:  
a) The issue date ($t = 0$) calculations are as follows:  
   1. An implied guaranteed interest rate (IGR) for the duration of the initial term,  
      is the guaranteed rate plus the accumulated option cost expressed as a percentage of the policy value to which the indexed benefit is applied. In turn, the accumulated option cost is the amount needed to provide the index-based benefit in excess of any other interest rate guarantee, accumulated to the end of the initial term at the appropriate maximum valuation rate.  
   2. An implied guaranteed rate for the period after the initial term.  
   3. The GMP, GMF, and valuation net premium based on the implied guaranteed rate.  

b) The valuation date ($t = t$) calculations are as follows:  
   1. The implied guaranteed rate for the remainder of the current period, using the option cost based on the market conditions at the valuation date.  
   2. The implied guaranteed rate for the period following the current period, based on the option cost on the valuation date.  
   3. A re-projection of future guaranteed benefits based on the implied valuation date.  
   4. The present value of the re-projected future guaranteed benefits.  

Note that the GMP, GMF, and valuation net premium remain the same as calculated at issue ($t = 0$).  

16.4.4 Contracts with Secondary Guarantees  
NAIC Actuarial Guideline 38 (AG 38) for reserves for UL products with secondary guarantees have 9 steps as follows:  

1. The minimum gross premium required to satisfy the secondary guarantees is derived at issue ($t = 0$) of the contract; the value of this premium will depend on whether the stipulated premium or the shadow fund method is in use. Its calculation uses the policy charges and credited interest rate guaranteed in the contract.  

2. The basic and deficiency reserves for the secondary guarantees are calculated using the minimum gross premium described in Step (1).  

3. The amount of actual contributions made in excess of the minimum gross premiums is determined, again with the process depending on whether the stipulated premium method or the shadow fund method is used.  

4. At the valuation date, $t$, a determination is made regarding amounts needed to fully fund the secondary guarantee.  
   (a) Under the shadow fund method, this would be the amount of the shadow fund account needed to fully fund the guarantee.  
   (b) Under contracts not using the shadow fund method, this would be the amount of cumulative premiums paid in excess of the required level such that no future premiums are required to fully fund the guarantee.  

Special rules apply to policies for which the secondary guarantee cannot be fully funded in advance. Here a prefunding ratio, $r$, ($r \leq 1$), is calculated that measures the level of prefunding for the secondary guarantee, and is eventually used in the calculation of reserves. It is defined as  

$$ r = \frac{\text{Excess Payment}}{\text{Net Single Premium Required to Fully Fund the Guarantee}}. $$  

(16.7)
5. At the valuation date, \( t \), the net single premium for the secondary guarantee coverage for the remainder of the secondary guarantee period is computed. \( \text{NSP}_t \).

6. A net amount of additional premiums is determined by multiplying the prefunding ratio described in Step (4) times the difference between the net single premium of Step (5) and the basic plus deficiency (if any) reserve of Step (2). \( r(\text{NSP}_t - \text{bdp}_t, V) \)

7. A reduced deficiency reserve is determined by multiplying the deficiency reserve (if any) by the complement of the pre-funding ratio from Step (4). \( \text{red}_d, V = (1 - r)_d, V \)

8. Then the actual reserve is the lesser of (a) the net single premium of Step (5), or (b) the amount in Step (6) plus the basic and deficiency (if any) reserve from Step (2). This result might be reduced by applicable policy surrender charges. \( \text{actual}_a, V = \min(\text{NSP}_t, \text{Step6} + \text{Step2}) \)

9. An increased basic reserve is computed by subtracting the reduced deficiency reserve of Step (7) from the reserve computed in Step (8), which then becomes the basic reserve. \( \text{incr}_b, V = \text{actual}_a, V - \text{red}_d, V \). Also \( AG_{38}, V = \text{actual}_a, V + \text{red}_d, V \)

ACTEX MLC Chapter 14 Study Manual Vol II Universal Life Insurance

14.1 Basic Policy Design

**Account Value Accumulation**

\[
AV_t = (AV_{t-1} + P_t - E_C_t - CoI_t) (1 + i_t)
\]

ULI with variable failure benefit (11.25c) \( AV_{t+1} = [AV_t + G_{t+1}(1 - r_{t+1}) - e_{t+1} - v_{t+1}q_{x+1}B](1 + i_{t+1}) \)

ULI with fixed failure benefit (11.26b) \( AV_{t+1} = [AV_t + G_{t+1}(1 - r_{t+1}) - e_{t+1}](1 + i_{t+1}) - q_{x+1}B(1 - AV_{t+1}) \)

(11.27) \( AV_{t+1} = \frac{[AV_t + G_{t+1}(1 - r_{t+1}) - e_{t+1}](1 + i_{t+1}) - q_{x+1}B}{p_{x+t}} \)

14.2 Cost of Insurance and Surrender Value

**Total Death Benefit**

| Specified Amount (Type A) | max \( (FA, f_t, AV_t) \) |
| Specified Amount plus the Account Value (Type B) | max \( (AV_t + X, f_t, AV_t) \) |

**Additional Death Benefit**

| Specified Amount (Type A) | \( ADB_t = \max (FA - AV_t, (f_t - 1) AV_t) \) |
| Specified Amount plus the Account Value (Type B) | \( ADB_t = \max (X, (f_t - 1) AV_t) \) |

**General Formula for the cost of Insurance (COI)**

\( CoI_t = q_t^i v_q \times ADB_t \)

where \( CoI_t \) is the cost of insurance for the \( t \)th time period, deducted from the account value at time \( t - 1 \), \( q_t^i \) is the death probability (for the \( t \)th time period) used to calculate the cost of insurance, \( v_q \) is the discount factor for discounting the cost of insurance to time \( t - 1 \), \( ADB_t \) is the additional death benefit at time \( t \).

**Cost of Insurance (COI)**

| Specified Amount (Type A) policy | \( CoI_t = \max (CoI_t^C, CoI_t^{w0, CF}) \) where \( CoI_t^C = \frac{q_t^i v_q (FA - (AV_{t-1} + P - EC_t)(1 + i_t))}{1 - q_t^i v_q (1 + i_t)(f_t - 1)} \) |

| Specified Amount plus the Account Value (Type B) policy | \( CoI_t^C = q_t^i v_q X \) and \( CoI_t^{CF} = \frac{q_t^i v_q (1 + i_t)(f_t - 1)(AV_{t-1} + P - EC_t)}{1 + q_t^i v_q (1 + i_t)(f_t - 1)} \) |

XXX14.5 Profit Testing

**Expected Profit**: \( Pr_t = (AV_{t-1} + P - EC_t)(1 + i_t) - EDB_t - ESB_t - EAV_t \)
17.2 Deferred Annuity Products

17.2.2 Variable Deferred Annuity

Investment Advisory Fee: \[ IAF_t(n) = FV_{t-1}(n) \cdot \left(1 + IAF_t(n)\right)^{1/365} - 1 \] \hspace{1cm} (17.1)

Net Investment Rate for day \( t \): \[ NIF_t(n) = \frac{NI_t(n) - IAF_t(n) + RCG_t(n) + UCG_t(n)}{FV_{t-1}(n)} \] \hspace{1cm} (17.2)

Net Investment Factor: \[ NIF_t(n) = 1 + NIR_t(n) \] \hspace{1cm} (17.3)

Sub-account \( n \) Fund Value recursion: \[ FV_t(n) = FV_{t-1}(n) \cdot NIF_t(n) - EXP_t(n) \] \hspace{1cm} (17.4)

Overall contract Account Value on day \( t \): \[ AV_t = \sum_{n} FV_t(n) \] \hspace{1cm} (17.5)

17.2.3 Equity-Indexed Deferred Annuity

a) point-to-point: \[ i_P = \frac{\text{Index value on closing day of index period}}{\text{Index value on initial day of index period}} - 1 \] \hspace{1cm} (17.6)

b) monthly ave: \[ i_{MA} = \frac{1}{12n} \left[ \text{Sum of index values on the last day of each month during index period} \right] \frac{\text{Index value on initial day of index period}}{\text{Index value on closing day of index period}} - 1 \] \hspace{1cm} (17.7)

c) with ratcheting \[ i_P = \frac{\text{Index value on closing day } t \text{ of index period}}{\text{Index value on day } t-1 \text{ of index period}} - 1 \] 

d) with ratcheting \[ i_{MA} = \frac{1}{12n} \left[ \text{Sum of index values on the last day of each month during index period} \right] \frac{\text{Index value on previous day } t-1 \text{ of index period}}{\text{Index value on closing day } t \text{ of index period}} - 1 \] 

17.3 Immediate Annuity Products

17.3.2 Variable Immediate Annuity \[ APU = \frac{P_t}{PUV_t} \] \hspace{1cm} (17.8)

\[ PUV_t = PUV_{t-1} \left(\frac{NIF_t}{1 + AIR_t}\right) \] \hspace{1cm} (17.9)

\[ P_t = (APU) (PUV_t) \] \hspace{1cm} (17.10)

\[ P_t = (APU) (PUV_{t-1}) \left(\frac{NIF_t}{1 + AIR_t}\right) = P_{t-1} \left(\frac{NIF_t}{1 + AIR_t}\right) \] \hspace{1cm} (17.11)