1. [10pts] Mark each of the following statements as true or false and justify your answer.

(a) \( \forall m \in \mathbb{N}, \exists n \in \mathbb{N}, m < n. \) True: for each \( m \) in \( \mathbb{N} \), take \( n = m + 1 \).

(b) \( \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, m < n. \) False: if \( m = 1 \), then \( n = 1 \) does not satisfy \( m < n \).

(c) \( \exists m \in \mathbb{N}, \forall n \in \mathbb{N}, m|n. \) True: if \( m = 1 \), then \( 1|n \) for each \( n \) in \( \mathbb{N} \).

(d) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy > x + 1. \) False: if \( x = 0 \), then there is no \( y \) in \( \mathbb{R} \) such that \( xy > x + 1 \).

2. [10pts] Let \( A = \{1, \{1\}\} \) and \( B = \mathcal{P}(A) \).

(a) Verify that \( A \cap B \in B \).

Solution. We have \( B = \{\emptyset, \{1\}, \{\{1\}\}, A\} \). Hence \( A \cap B = \{\{1\}\}, \) which is an element of \( B \).

(b) Find \( |\mathcal{P}(A \times B)| \).

Solution. \( |A \times B| = |A| \times |B| = 2 \times 4 = 8 \). Hence \( |\mathcal{P}(A \times B)| = 2^8 \).

3. [10pts] (a) Let \( A = \{1, 2, 3\} \). Give an example of a subset \( B \) of \( A \) such that

\[
(A \times A) - (B \times B) \neq (A - B) \times (A - B).
\]

Solution. Let \( B = \{1\} \) and let \( a = (1, 2) \). Then \( a \in A \times A \) but \( a \notin B \times B \) (\( : 2 \notin B \)), so \( a \in (A \times A) - (B \times B) \). However, \( 1 \notin A - B \), so \( a \notin (A - B) \times (A - B) \). This shows that \( (A \times A) - (B \times B) \neq (A - B) \times (A - B) \).

(b) Prove that if \( C, D \) are sets, then \( (C \times D) \cap (D \times C) = (C \cap D) \times (C \cap D) \).

Proof. Let \( (x, y) \in (C \times D) \cap (D \times C) \). Then \( (x, y) \in C \times D \) and \( (x, y) \in D \times C \), so \( x \in C \cap D \) and \( y \in C \cap D \), i.e. \( (x, y) \in (C \cap D) \times (C \cap D) \). This proves \( (C \times D) \cap (D \times C) \subseteq (C \cap D) \times (C \cap D) \). For the reverse inclusion, let \( (x, y) \in (C \cap D) \times (C \cap D) \). Then \( x \in C \cap D \) and \( y \in C \cap D \), so that \( (x, y) \in C \times D \) and \( (x, y) \in D \times C \), i.e. \( (x, y) \in (C \times D) \cap (D \times C) \). This proves \( (C \cap D) \times (C \cap D) \subseteq (C \times D) \cap (D \times C) \) and the proof is complete.

4. [10pts] Let \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \). Prove the following statements.

(a) If \( |1 + n| + |1 - n| \leq 1 \), then \( |n^4 - 1| < 16 \).

Proof. Since \( n \geq 1 \), \( |1 + n| \geq 2 \), hence \( |1 + n| + |1 - n| \geq 2 \). This means that the statement \(|1 + n| + |1 - n| \leq 1\) is false and so the implication to prove is vacuously true.

(b) If \( x^2 > 4 \), then \( |1 + x| + |1 - x| \geq 2 \).

Proof. By the triangle inequality, \(|1 + x| + |1 - x| \geq |(1 + x) + (1 - x)| = 2 \). Hence the statement \(|1 + x| + |1 - x| \geq 2\) is true and so the implication to prove is trivially true.
5. [10pts] Let $x, y$ be integers. Prove the following statements.

(a) If $x$ and $y$ are odd, then $4 \mid ((x + y)^2 + (x - y)^2)$.

**Proof.** Let $x = 2h + 1$, $y = 2k + 1$ ($h, k \in \mathbb{Z}$). Then $(x+y)^2 + (x-y)^2 = 2(x^2 + y^2) = 2(4h^2 + 4h + 1 + 4k^2 + 4k + 1) = 4(2h^2 + 2h + 2k^2 + 2k + 1)$. Since $2h^2 + 2h + 2k^2 + 2k + 1 \in \mathbb{Z}$, it follows that $4 \mid ((x + y)^2 + (x - y)^2)$. ■

(b) If $4 \mid (x^2 + y^2)$, then $x$ and $y$ are both odd or both even.

**Proof.**

- We can use a direct proof:

  Suppose $4 \mid (x^2 + y^2)$, then $x^2 + y^2 = 4k$ for some integer $k$. Hence $x^2 = 2(2k - y^2)$, which is even. Therefore $x$ must be even, say $x = 2h$ for some integer $h$. We then get $4h^2 + y^2 = 4k$, i.e. $y^2 = 2(k - 2h^2)$, which is even. Hence $y^2$ is even and so $y$ is even as well. This means $x$ and $y$ are both even and so they have the same parity.

- We can also use the contrapositive:

  Suppose $x$ and $y$ have opposite parity. We have two cases:
  
  (i) Case $x$ is even and $y$ is odd: In this case $x = 2h$, $y = 2k+1$ for some integers $h, k$. Hence $x^2 + 2y^2 = 4h^2 + 2(4k^2 + 4k + 1) = 2(2h^2 + 4k^2 + 4k + 1)$, and since $2h^2 + 4k^2 + 4k + 1 = 2(h^2 + 2k^2 + 2k) + 1$ is odd (as $h^2 + 2k^2 + 2k \in \mathbb{Z}$), we deduce that $x^2 + 2y^2$ is not divisible by 4 (but it is even).
  
  (ii) Case $x$ is odd and $y$ is even: In this case $x = 2h + 1$, $y = 2k$ for some integers $h, k$. Hence $x^2 + 2y^2 = 4(h^2 + h + k^2) + 1$, which clearly is odd (as $h^2 + h + k^2 \in \mathbb{Z}$) so that $4 \nmid (x^2 + y^2)$. ■

---

6. [10pts] Let $a, b \in \mathbb{Z}$. Prove the following statements.

(a) If $a \equiv 1 \pmod{2}$ and $b \equiv 2 \pmod{3}$, then $3a - 2b \equiv 5 \pmod{6}$

**Proof.** $3a \equiv 3 \pmod{6}$ and $2b \equiv 4 \pmod{6}$, so $3a - 2b \equiv -1 \equiv 5 \pmod{6}$. ■

(b) If $a \not\equiv 2 \pmod{4}$, then $a^3 \equiv a \pmod{4}$.

**Proof.** We have 3 cases.

Case $a \equiv 0 \pmod{4}$: then $a^3 \equiv 0 \equiv a \pmod{4}$.

Case $a \equiv 1 \pmod{4}$: then $a^3 \equiv 1 \equiv a \pmod{4}$.

Case $a \equiv 3 \pmod{4}$: then $a \equiv -1 \pmod{4}$, so $a^3 \equiv -1 \equiv a \pmod{4}$. ■