King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 302, Semester 181 (2018-2019)

EXAM I
October 10, 2018

Allowed Time: 120 mins

Student Name:
Student ID Number:
Section Number:

Instructions:
1. Write neatly and legibly — you may lose points for messy work.
2. Show all your work — no points for answers without justification.
3. Programmable calculators and Mobiles are not allowed.
4. Make sure that you have 7 problems (7 pages + cover page + Scratch sheet).

<table>
<thead>
<tr>
<th>Problem No.</th>
<th>Points</th>
<th>Out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Total:</td>
<td></td>
<td>100</td>
</tr>
</tbody>
</table>
Problem 1. a. Check if the vectors $U = (1, 1, 0)$, $V = (1, 1, 2)$, and $W = (2, 2, 1)$ are linearly dependent or independent?

\[ \alpha U + \beta V + \gamma W = 0 \iff \begin{cases} \langle \alpha, \alpha, 0 \rangle + \langle \beta, \beta, 2\beta \rangle + \langle 2\gamma, 2\gamma, \gamma \rangle = 0 \iff \\ \langle \alpha + \beta + 2\gamma, \alpha + \beta + 2\gamma, 2\beta + \gamma \rangle = \langle 0, 0, 0 \rangle \iff \\ \alpha + \beta + 2\gamma = 0 \\ 2\beta + \gamma = 0 \Rightarrow 2\alpha + 2\beta + 4\gamma - (2\beta + \gamma) = 0 \\ \Rightarrow 2\alpha + 3\gamma = 0 \Rightarrow \alpha = -\frac{3\gamma}{2} \\ \end{cases} \]

Also, $2\beta + \gamma = 0 \Rightarrow \beta = -\frac{\gamma}{2}$

So, we can find $\gamma = 2$, $\beta = -1$, $\alpha = -3$ so that $-3U - V + 2W = 0$

\[ \therefore U, V, W \text{ are linearly dependent.} \]

b. Show that $S = \{X = (x, y, 0, z) \in \mathbb{R}^4 : xyz \geq 0 \}$ is not a subspace of $\mathbb{R}^4$.

$U = (1, 1, 0, 1) \in S$ but $-U = (-1, -1, 0, -1) \not\in S$

Since $(-1)(-1)(-1) < 0$. Thus $S$ is not a subspace.
Problem 2. According to the values of $\alpha$, find the rank of the matrix

$$A = \begin{pmatrix} -1 & 2 & -3 \\ 1 & -2 & \alpha \\ 0 & -3 & 0 \\ -1 & -1 & -3 \end{pmatrix}$$

So,

$$(-1 \ 2 \ -3) \quad R_2 + R_1 \quad \rightarrow \quad \begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & \alpha - 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_4 - R_3$$

$$\begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & \alpha - 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad R_3 \leftrightarrow R_4 \quad \begin{pmatrix} -1 & 2 & -3 \\ 0 & 0 & \alpha - 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So, if $\alpha = 3$ then $\text{rank } A = 2$

if $\alpha \neq 3$ then $\text{rank } A = 3$
Problem 3. Use the Gauss method to find all solutions of the system

\[
\begin{align*}
-3x + 2y - 6z &= 6 \\
5x + 7y - 5z &= 6 \\
x + 4y - 2z &= 8
\end{align*}
\]

So,

\[
\begin{pmatrix}
-3 & 2 & -6 & | & 6 \\
5 & 7 & -5 & | & 6 \\
1 & 4 & -2 & | & 8
\end{pmatrix}
\xrightarrow{R_1 \leftrightarrow R_3}
\begin{pmatrix}
1 & 4 & -2 & | & 8 \\
5 & 7 & -5 & | & 6 \\
3 & 2 & -6 & | & 8
\end{pmatrix}
\xrightarrow{R_3 + 3R_1}
\begin{pmatrix}
1 & 4 & -2 & | & 8 \\
0 & -13 & 5 & | & 34 \\
0 & 14 & -12 & | & 30
\end{pmatrix}
\xrightarrow{R_2 + R_3}
\begin{pmatrix}
1 & 4 & -2 & | & 8 \\
0 & 1 & -7 & | & 4 \\
0 & 14 & -12 & | & 30
\end{pmatrix}
\xrightarrow{R_3 / 3}
\begin{pmatrix}
1 & 4 & -2 & | & 8 \\
0 & 1 & -7 & | & 4 \\
0 & 0 & 43 & | & 10
\end{pmatrix}
\xrightarrow{R_1 - 4R_2}
\begin{pmatrix}
1 & 0 & 26 & | & 24 \\
0 & 1 & -7 & | & 4 \\
0 & 0 & 43 & | & 10
\end{pmatrix}
\xrightarrow{R_3 / 43}
\begin{pmatrix}
1 & 0 & 26 & | & 24 \\
0 & 1 & -7 & | & 4 \\
0 & 0 & 1 & | & 1
\end{pmatrix}
\xrightarrow{R_2 + 7R_3}
\begin{pmatrix}
1 & 0 & 0 & | & -2 \\
0 & 1 & 0 & | & 3 \\
0 & 0 & 1 & | & 1
\end{pmatrix}
\Rightarrow \text{the solution is}

\[
X = \begin{pmatrix}
-2 \\
3 \\
1
\end{pmatrix}.
\]
Problem 4. Use elementary operations to find the inverse of

\[ A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} \]

**Sol.**

\[
\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[ R_2 - 2R_1 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -2 & -1 \end{pmatrix} \]

\[ R_3 - R_1 \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -2 & -1 \end{pmatrix} \]

\[ R_1 - 2R_2 \rightarrow \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ R_3 + 2R_2 \rightarrow \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ -R_3 \rightarrow \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ R_1 - 9R_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ R_2 + 3R_3 \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \rightarrow \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \]

So, \[ A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \]
Problem 5.

a. Find all the eigenvalues of

\[ A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 6 \\ -3 & 0 & 7-\lambda \end{pmatrix} \]

\[ \begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & 5-\lambda & 6 \\ -3 & 0 & 7-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 5-\lambda & 6 \\ 0 & 7-\lambda \end{vmatrix} + 3 \begin{vmatrix} 0 & 5-\lambda \\ -3 & 0 \end{vmatrix} \]

\[ = (1-\lambda)(5-\lambda)(7-\lambda) + 9(5-\lambda) \]

\[ = (5-\lambda)(\lambda^2 - 8\lambda + 7 + 9) \]

\[ = (5-\lambda)(\lambda - 4)^2 = 0 \]

\[ \Rightarrow \lambda = 4, 4, 5 \text{ are the eigenvalues.} \]

a. Is A diagonalizable and why?

We find eigenvector(s) of \( \lambda = 4 \) since it is repeated

\[ \begin{pmatrix} -3 & 0 & 3 \\ 0 & 1 & 6 \\ -3 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \\ -3 & 0 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 6 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \Rightarrow x = \frac{3}{8}, \quad y = -6z. \]

So, an eigenvector is

\[ \begin{pmatrix} 1 \\ -6 \\ 1 \end{pmatrix} \]

4 is repeated and we only got an eigenvector

\[ \Rightarrow A \text{ is not diagonalizable.} \]
Problem 6. Given the matrix \( A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} \). Find a nonsingular matrix \( P \) and a diagonal matrix \( D \) such that \( P^{-1}AP = D \). Compute \( A^{21} \).

So\( l. \begin{vmatrix} 3 - \lambda & 2 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0 \)

\( \Rightarrow \) \( \lambda = 1, 2 \) are the eigenvalues.

Eigenvalues
\( \lambda = 1 \)
\[
\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{R_{1/2}} \rightarrow \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}
\]

\( \Rightarrow \) \( X = -y \) \( \Rightarrow \) \( \mathbf{k}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) is an eigenvector.

\( \lambda = 2 \)
\[
\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}^{R_{2}+R_{1}} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}
\]

\( \Rightarrow \) \( X = -2y \) \( \Rightarrow \) \( \mathbf{k}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) is an eigenvector.

\( P = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \)

We have \( P^{-1} = PD \quad \Rightarrow \quad A^{21} = PD^{21}P^{-1} \)

\[
A^{21} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^{21} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 - 2^{22} \\ -1 \\ 2^{24} \\ -1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 + 2^{22} & -2 + 2^{22} \\ 1 - 2^{24} & 2 - 2^{24} \end{pmatrix}
\]
Problem 7.

a) Find all the values of $k$ for which the matrix

$$A = k \begin{pmatrix} 1+\sqrt{2} & \sqrt{2} & \sqrt{2} - 1 \\ -\sqrt{2} & 2 & \sqrt{2} \\ -1+\sqrt{2} & -\sqrt{2} & 1+\sqrt{2} \end{pmatrix}$$

is orthogonal

$$\langle 1+\sqrt{2}, \sqrt{2}, \sqrt{2} - 1 \rangle \cdot \langle -\sqrt{2}, 2, \sqrt{2} \rangle = -\sqrt{2} \cdot 2 + 2\sqrt{2} + 2 - \sqrt{2} = 0$$

$$\langle 1+\sqrt{2}, \sqrt{2}, \sqrt{2} - 1 \rangle \cdot \langle 1+\sqrt{2}, -\sqrt{2}, 1+\sqrt{2} \rangle = 1 - 2 + 1 = 0$$

$$\langle -\sqrt{2}, 2, \sqrt{2} \rangle \cdot \langle -1+\sqrt{2}, -\sqrt{2}, 1+\sqrt{2} \rangle = \sqrt{2} - 2 - 2\sqrt{2} + \sqrt{2} + 2 = 0$$

The rows are mutually orthogonal.

We compute the magnitude of each row:

$$\| k \langle 1+\sqrt{2}, \sqrt{2}, \sqrt{2} - 1 \rangle \|^2 = 8k^2 \quad \text{(Similarly for all)}$$

$$8k^2 = 1 \iff k = \pm \frac{\sqrt{2}}{4}$$

For these values $A$ is orthogonal.

b. Let

$$M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$$

Show that $M$ is orthogonal if and only if $\sigma = 0$ and $S = \pm 1$.

$$\langle a, b, c \rangle \cdot \langle c, a, b \rangle = ab + ba + cb = \sigma$$

Similarly for $\langle a, b, c \rangle \cdot \langle b, c, a \rangle = \sigma$

and $\langle c, a, b \rangle \cdot \langle b, c, a \rangle = \sigma$

The magnitude of each column or row is $\sqrt{a^2 + b^2 + c^2}$

So $M$ is orthogonal iff $\sigma = 0$ and $a^2 + b^2 + c^2 = 1$
\[ S^2 = (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \]
\[ = a^2 + b^2 + c^2 + 2bc = 1 + 0 = 1. \]

\[ \implies S = \pm 1. \]