Q:1 (10 points) Find the parametric equations of the tangent line of the vector function $\mathbf{r}(t) = < t^2,\ 2\sin(t),\ 2\cos(t) >$ at $t = \frac{\pi}{3}$.

Sol.: $r'(t) = \langle 2t, 2\cos t, -2\sin t \rangle$

$r'(\frac{\pi}{3}) = \langle \frac{2\pi}{3}, 1, -\sqrt{3} \rangle$

$r(\frac{\pi}{3}) = \langle \frac{\pi^2}{9}, \sqrt{3}, 1 \rangle$

Parametric equations:

$x = \frac{\pi^2}{9} + \frac{2\pi}{3} t$

$y = \sqrt{3} + t$

$z = 1 - \sqrt{3}\, t$

Q:2 (8 + 4 = 12 points) (a) Find the directional derivative of $f(x,y) = x^2 + y^2$ at $(3,4)$ in the direction of a tangent vector to the graph of $2x^2 + y^2 = 9$ at $(2,1)$.

Sol: $2x^2 + y^2 = 9$

Implicit differentiation w.r. to x

$$4x + 2yy' = 0$$

$$\Rightarrow y' = -\frac{2x}{y}$$

The slope of tangent line at $(2,1)$ is

$$y'\Big|_{(2,1)} = -\frac{4}{1}$$

Vector $= \pm \langle 1, -4 \rangle$

unit vector $\vec{u} = \pm \frac{\langle 1, -4 \rangle}{\sqrt{17}}$

$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla f(3,4) = \langle 6, 8 \rangle$

$D_{\vec{u}} f(3,4) = \nabla f \cdot \vec{u} = \pm \langle 6,8 \rangle \cdot \frac{\langle 1,-4 \rangle}{\sqrt{17}}$

$$= \pm \frac{26}{\sqrt{17}}$$

(b) Suppose $\nabla f(a,b) = \langle 6, 8 \rangle$. Find a unit vector $\mathbf{u}$ so that $D_{\mathbf{u}}f(a,b) = 0$.

Sol: $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$

$0 = \langle 6, 8 \rangle \cdot \langle u_1, u_2 \rangle$

$\Rightarrow 6u_1 + 8u_2 = 0.$ —(1)

Since $\vec{u}$ is unit vector, therefore, $u_1^2 + u_2^2 = 1$ —(2)

(1) $\Rightarrow u_1 = -\frac{4}{3}u_2$

(2) $\Rightarrow \frac{16}{9}u_2^2 + u_2^2 = 1 \Rightarrow u_2 = \pm \frac{3}{5}$

$u_1 = \mp \frac{4}{5}$

$\vec{u} = \mp \langle \frac{4}{5}, -\frac{3}{5} \rangle$

Q:3 (12 points) Verify that div(**F** × **G**) = **G** · curl **F** − **F** · curl **G**.

Sol.: Let $\vec{F} = \langle f_1, f_2, f_3 \rangle$ and $G = \langle g_1, g_2, g_3 \rangle$

$$\vec{F} \times \bar{G} = \begin{vmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= \langle f_2 g_3 - f_3 g_2, \ -f_1 g_3 + f_3 g_1, \ f_1 g_2 - f_2 g_1 \rangle$$

$$\operatorname{div}(\vec{F} \times \bar{G}) = \frac{\partial f_2}{\partial x} g_3 + f_2 \frac{\partial g_3}{\partial x} - \frac{\partial f_3}{\partial x} g_2 - f_3 \frac{\partial g}{\partial x}$$
$$- \frac{\partial f_1}{\partial y} g_3 - f_1 \frac{\partial g_3}{\partial y} + \frac{\partial f_3}{\partial y} g_1 + f_3 \frac{\partial g_1}{\partial x}$$
$$+ \frac{\partial f_1}{\partial z} g_2 + f_1 \frac{\partial g_2}{\partial z} - \frac{\partial f_2}{\partial z} g_1 - f_2 \frac{\partial g_1}{\partial z}$$

Now.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left\langle \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, -\frac{\partial f_3}{\partial x} + \frac{\partial f_1}{\partial z}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right\rangle$$

$$\vec{G} \cdot \operatorname{curl} \vec{F} = \frac{\partial f_3}{\partial y} g_1 - \frac{\partial f_2}{\partial z} g_1 - \frac{\partial f_3}{\partial x} g_2 + \frac{\partial f_1}{\partial z} g_2 + \frac{\partial f_2}{\partial x} g_3 - \frac{\partial f_1}{\partial y} g_3$$

Similarly

$$\vec{F} \cdot \operatorname{curl} \bar{G} = \frac{\partial g_3}{\partial y} f_1 - \frac{\partial g_2}{\partial z} f_2 - \frac{\partial g_3}{\partial x} f_2 + \frac{\partial g_1}{\partial z} f_2 + \frac{\partial g_2}{\partial x} f_3 - \frac{\partial g_1}{\partial y} f_3$$

$$\bar{G} \cdot \operatorname{curl} \vec{F} - \vec{F} \cdot \operatorname{curl} \bar{G} = \frac{\partial f_3}{\partial y} g_1 - \frac{\partial f_2}{\partial z} g_1 - \frac{\partial f_3}{\partial x} g_2 + \frac{\partial f_1}{\partial z} g_2$$
$$+ \frac{\partial f_2}{\partial x} g_3 - \frac{\partial f_1}{\partial y} g_3 - \frac{\partial g_3}{\partial y} f_1 + \frac{\partial g_2}{\partial z} f_2$$
$$+ \frac{\partial g_3}{\partial x} f_2 - \frac{\partial g_1}{\partial z} f_2 - \frac{\partial g_2}{\partial x} f_3 + \frac{\partial g_1}{\partial y} f_3$$

$$= \operatorname{div}(\vec{F} \times \bar{G})$$

Q:4 (16 points) Let $\mathbf{F} = < (y + yz), (x + 3z^3 + xz), (9\,yz^2 + xy - 1) >$ be the vector field on a certain region of space.

(a) Verify that $\mathbf{F}$ is a conservative vector field.

(b) Find a potential function for $\mathbf{F}$.

(c) Use the Fundamental theorem and potential function to evaluate $\int_{(1,1,-1)}^{(1,2,1)} \mathbf{F} \bullet dr$.

(a)
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+yz & x+3z^3+xz & 9yz^2+xy-1 \end{vmatrix}$$

$$= \langle 9z^2+x-9z^2-x,\ -y+y,\ 1+z-1-z \rangle = \vec{0}$$

$\Rightarrow \vec{F}$ is a conservative vector field.

(b) $\vec{F} = \nabla \phi \Rightarrow \frac{\partial \phi}{\partial x} = y+yz$, $\frac{\partial \phi}{\partial y} = x+3z^3+xz$, $\frac{\partial \phi}{\partial z} = 9yz^2+xy-1$

Int. w.r. to x, we get $\phi = xy + xyz + G(y,z)$

$$\frac{\partial \phi}{\partial y} = x + xz + \frac{\partial G}{\partial y} = x + 3z^3 + xz$$

$$\Rightarrow \frac{\partial G}{\partial y} = 3z^3$$

$$\Rightarrow G = 3yz^3 + h(z)$$

$$\phi = xy + xyz + 3yz^3 + h(z)$$

$$\frac{\partial \phi}{\partial z} = xy + 9yz^2 + h'(z) = 9yz^2 + xy - 1$$

$$\Rightarrow h'(z) = -1 \quad \Rightarrow h(z) = -z + C$$

$$\therefore \phi = xy + xyz + 3yz^3 - z$$

Fundamental theorem
$$\int_{(1,1,-1)}^{(1,2,1)} \vec{F} \cdot d\vec{r} = \phi(1,2,1) - \phi(1,1,-1)$$

$$= (2+2+6-1) - (1-1-3+1)$$

$$= 9 + 2$$

$$= 11$$

Q:5 (8 + 6 = 14 points) (a) Use Green's theorem to evaluate $\int_C xy^2dx - x^2ydy$, where C consists of the boundary of the region in the first quadrant that is bounded by the graph of $1 \le x^2 + y^2 \le 4$.

(a) Green's theorem $\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$

$P = xy^2$ $\quad Q = -x^2y$

$P_y = 2xy$ $\quad Q_x = -2xy$

RHS $\iint_R -4xy\,dA = \int_{\theta=0}^{\frac{\pi}{2}} \int_1^2 -4r^2 \sin\theta\cos\theta\, r\,dr\,d\theta$

$= -4 \int_0^{\pi/2} \int_1^2 r^3 \sin\theta\cos\theta\,dr\,d\theta$

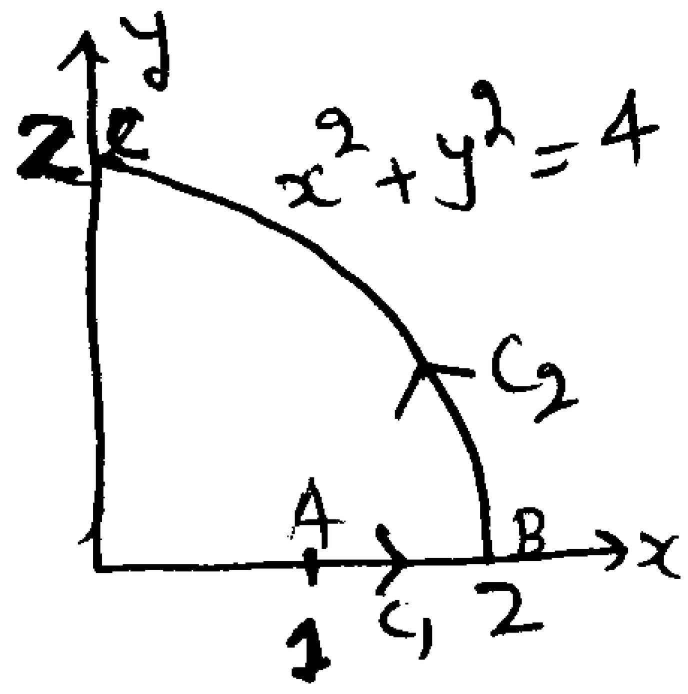
$= -4 \left[\frac{r^4}{4}\right]_1^2 \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta$

$= -4\left[4 - \frac{1}{4}\right] \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$

$= -4 \cdot \frac{15}{4} \cdot \frac{1}{2}$

$= -\frac{15}{2}$

(b) Compute $\int_C xy^2dx - x^2ydy$ by parameterizing the path in the figure.



$C_1 : \vec{r}(t) = \langle x, y\rangle = \langle t, 0\rangle,\ 1 \le t \le 2$

$\int_{C_1} t \cdot 0 \cdot dt - 0 = 0$

$C_2 : \vec{r}(t) = \langle 2\cos t, 2\sin t\rangle,\ 0 \le t \le \pi/2$

$dx = -2\sin t\,dt,\ dy = 2\cos t\,dt$

$\int_{C_2} = \int_0^{\pi/2} \left[(2\cos t \cdot 4\sin^2 t)(-2\sin t) - 4\cos^2 t \cdot 2\sin t \cdot 2\cos t\right] dt$

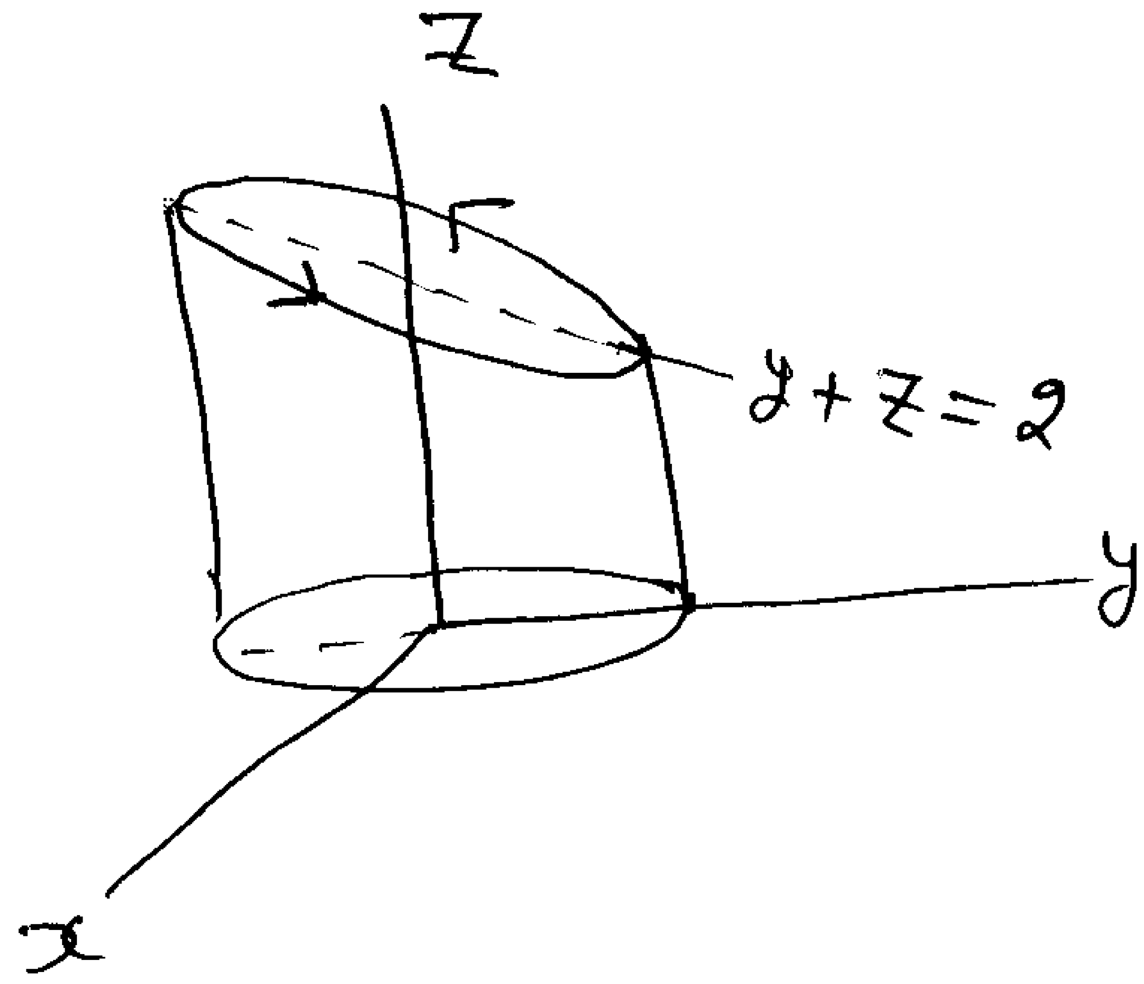
$= \int_0^{\pi/2} -16\cos t\sin t\,dt = -8\int_0^{\pi/2} \sin 2t\,dt = 4\left[\cos 2t\right]_0^{\pi/2}$

$= 4[\cos\pi - \cos 0] = 4[-1-1] = -8$

$\therefore \int_C = \int_{C_1} + \int_{C_2} = -8$

Q:6 (16 points) Use Stokes' theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = < -y^2, x, z^2 >$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (orient C to be counterclockwise when viewed from above)

Sol: Stokes' theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds$



$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$$= \langle 0, 0, 1+2y \rangle$$

$$g(x,y,z) = z + y - 2$$

$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 0, 1, 1 \rangle}{\sqrt{2}}$$

$$dS = \sqrt{2}\, dA$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$= \iint_R (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + \frac{2}{3} r^3 \sin\theta \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3}\sin\theta \right] d\theta$$

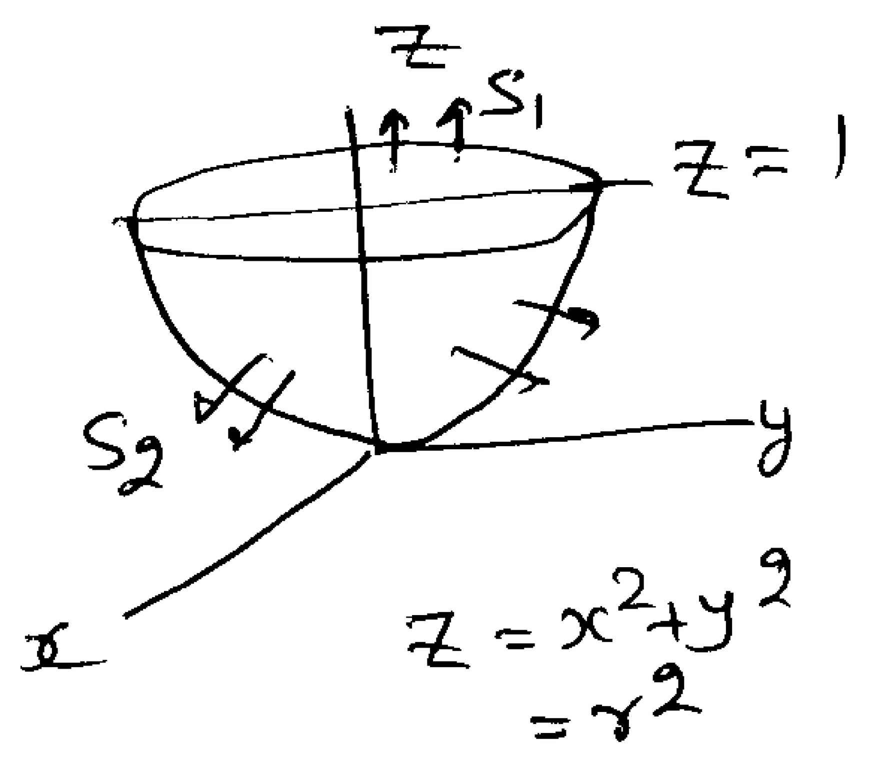
$$= \left. \frac{1}{2}\theta - \frac{2}{3}\cos\theta \right]_0^{2\pi}$$

$$= \pi - \frac{2}{3} + \frac{2}{3}$$

$$= \pi$$

Q:7 (10 + 10 = 20 points) Verify divergence theorem by evaluating **BOTH** integrals (**BOTH** sides of the identity in the divergence theorem) with $\mathbf{F}(\mathbf{x},\mathbf{y},\mathbf{z}) = < y,\ x,\ z^2 >$ and D is the region in R^3 bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

Sol: Divergence theorem $\iint_S (\vec{F}\cdot\vec{n})\,dS = \iiint_D \nabla\cdot\vec{F}\,dV$



LHS. $\iint_S = \iint_{S_1} + \iint_{S_2}$

on S_1: $\vec{n} = \langle 0,0,1\rangle$, $z=1$; $dS = dA$

$$\iint_{S_1}(\vec{F}\cdot\vec{n})\,dS = \iint z^2\,dA$$

$$= \iint_R dA = \pi$$

on S_2: $g(x,y,z) = x^2+y^2-z$.

$$\vec{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 2x, 2y, -1\rangle}{\sqrt{1+4x^2+4y^2}} \quad ; \quad dS = \sqrt{1+4(x^2+y^2)}$$

$$\iint_{S_2} = \iint_R (4xy - z^2)\,dA = \iint_R (4r^2\cos\theta\sin\theta - r^4)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_0^1 [4r^3\cos\theta\sin\theta - r^5]\,dr\,d\theta$$

$$= \left[4\frac{r^4}{4}\right]_0^1 \left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} - \left[\frac{r^6}{6}\right]_0^1\cdot 2\pi = -\frac{\pi}{3}$$

$$\therefore \boxed{LHS = \pi - \frac{\pi}{3} = \frac{2\pi}{3}}$$

RHS:

$$\nabla\cdot\vec{F} = 0 + 0 + 2z$$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D 2z\cdot r\,dr\,d\theta\,dz$$

$$= \int\int\int_{z=r^2}^1 2z\,dz\,.\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^1 (1-r^4)\,.\,r\,dr\,d\theta = 2\pi\left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1$$

$$= 2\pi\left[\frac{1}{2} - \frac{1}{6}\right] = \frac{2\pi}{3}.$$