King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 535
Functional Analysis

31 October 2018

Name: ____

This exam contains 8 pages (including this cover page) and 7 questions. Total of points is 100.

Distribution of Marks

Question	Points	Score
1	18	
2	20	
3	18	
4	14	
5	16	
6	7	
7	7	
Total:	100	

1. (a) (10 points) Let U be a vector space et let $f \in U^*$ be a non-zero linear functional. Show that there exists a vector $u \in U$ such that $f(u) = 1$.

   (b) (8 points) Let U be a linear vector space and let V be a proper subspace of U. We pick a vector $u_0 \in U \setminus V$. Show that there is a linear functional $f_0 \in U^*$ such that $f_0(u_0) = 1$ and $f_0(u) = 0$ for all $u \in V$.

2. (a) (10 points) Let $f : (\mathbb{R}^n, ||.||_2) \to \mathbb{R}$ defined by

$$f(x_1, \ldots, x_n) =< a, x >= \sum_{i=1}^{n} a_i x_i$$

Show that f is linear and find $||f||$.

(b) (10 points) Let $F : (\mathcal{C}[a, b], ||.||_\infty) \to \mathbb{R}$ defined by $F(x) = \int_a^b x(t)dt$
Find $||F||$.

3. (a) (10 points) Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Show that T is continuous if and only if T is bounded.

(b) (8 points) Let $A, B \subset X$ are non-empty subspaces of a metric space (X, d). Let us assume that A is compact and B is closed. Show that $d(A, B) = 0$ if and only if $A \cap B \neq \emptyset$. Give an example of A, B are closed and $A \cap B = \emptyset$ but $d(A, B) = 0$.

4. (a) (6 points) Let U and V be two normed linear spaces with $\dim U < +\infty$. Show that any linear operator $A : U \to V$ is bounded and there exists $u \in U \setminus \{0\}$ such that $||A(u)|| = ||A||.||u||$

(b) (8 points) Let U be a normed space and V be a Banach space.
Prove that $\mathcal{B}(U, V)$ the set of all continuous linear operators from U to V is a Banach space equipped with the norm $||T|| = \sup_{x \in U, x \neq 0} \dfrac{||T(x)||}{||x||}$.

5. (a) (8 points) Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction mapping with a constant k. If $x^* \in X$ is the fixed point of f, then show that

$$d(x^*, x) \leq \frac{1}{1-k} d(x, f(x))$$

for all $x \in X$.

(b) (8 points) Let $T : [1, +\infty) \to [1, +\infty)$ be given by

$$T(x) = \frac{2}{3}(x + \frac{1}{x}).$$

Show that T is a contraction.

6. (a) (7 points) Consider the linear functional $f : l_1(\mathbb{N}) \to \mathbb{R}$ defined by

$$f(u) = \sum_{n=1}^{\infty}(1 - \frac{1}{n})u_n$$

for all $u = (u_n) \in l_1(\mathbb{N})$.

Show that $||f|| = 1$.

7. (a) (7 points) Consider the sequence of linear operator $T_n : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ defined by

$$T_n(x) = (0, 0, \ldots, x_{n+1}, x_{n+2}, \ldots).$$

Show that $T_n(x)$ converges to zero in $l_2(\mathbb{N})$, for all $x \in l_2(\mathbb{N})$, but $||T_n||$ does not converge to zero.
Hint: show that $||T_n|| = 1$ by considering $e_{n+1} = (0, 0, \ldots, 1, 0, \ldots)$.