Name: ____________________________

This exam contains 11 pages (including this cover page) and 10 questions. Total of points is 130.

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1. (a) (10 points) Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$. Show that the following are equivalent
   1. $u \in \mathcal{M}^\perp$.
   2. $||u - v|| \geq ||u||$ for all $v \in \mathcal{M}$.

(b) (5 points) Let $\mathcal{H}$ be an inner product space. For a non-zero vector $a \in \mathcal{H}$. Set

$$\mathcal{M} = \{u \in \mathcal{H} : (u, a) = 0\}.$$ 

Find $\mathcal{M}^\perp$. 

2. (a) (10 points) Let \{\phi_1, \ldots, \phi_n\} be an orthonormal set of vectors in a Hilbert space \( \mathcal{H} \) and \( \mathcal{M} = \langle \phi_1, \ldots, \phi_n \rangle \). Let \( x \in \mathcal{H} \) and set \( y = \sum_{i=1}^{n} (x, \phi_i) \phi_i \). Show that

1. \( y \) is orthogonal to \( x - y \).
2. \( x - y \in \mathcal{M}^\perp \)
3. \( d(x, \mathcal{M}) = ||x - y|| \)

(b) (8 points) Let \( \mathcal{M} \) be a one-dimensional subspace of \( \mathcal{H} \). Let \( a \) be a non-zero element of \( \mathcal{M} \). Show that

\[
d(x, \mathcal{M}^\perp) = \frac{|\langle x, a \rangle|}{||a||}
\]
3. (a) (15 points) Let \( H \) be an inner product space and let \( p : H \to H \) be a projection

1. Show that \( H = \mathcal{R}(p) \oplus \text{Ker} \, p \).

2. Recall that \( p \) is called an \textbf{orthogonal projection} if its null-space and its range are orthogonal.

   Show that if \( p \) is an orthogonal projection, then

   \[
   \text{Ker} \, p = \mathcal{R}(p)^\perp \quad \text{and} \quad \mathcal{R}(p) = (\text{Ker} \, p)^\perp.
   \]

3. Let \( p : H \to H \) be a non-zero orthogonal projection. Show that \( p \) is continuous and \( \|p\| = 1 \).
4. (a) (10 points) Let $\mathcal{M}$ be a non-empty, closed subspace of a Hilbert space $\mathcal{H}$.

1. Show that there exists a unique projection $p_\mathcal{M} : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{R}(p_\mathcal{M}) = \mathcal{M}$.
2. Show that $d(u, \mathcal{M}) = ||u - p_\mathcal{M}(u)||$. 
5. (a) (7 points) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Prove that for any $f \in \mathcal{M}'$, the functional $F: \mathcal{H} \to \mathbb{R}$ defined by

$$F(u) = f(\text{pr}_\mathcal{M}(u))$$

is an extension of $f$ with $||F|| = ||f||$. 


6. (a) (10 points) Let $f$ be a continuous linear functional on a Hilbert space $\mathcal{H}$. Prove that there exists a unique $z \in \mathcal{H}$ such that $f(x) = \langle x, z \rangle$ and $\|f\| = \|z\|.$

(b) (5 points) Let $\{\phi_n\}$ be an orthonormal basis of $\mathcal{H}$, show that

$$z = \sum_{n=1}^{\infty} f(\phi_n) \phi_n.$$
7. (a) (10 points) Let $\mathcal{H}$ be a Hilbert space and $\{\phi_n\}$ be an orthonormal system. Prove that $\phi_n$ converges weakly to zero, that is, $\langle y, \phi_n \rangle \rightarrow 0$ for all $y \in \mathcal{H}$. 


8.  (a) (10 points) Let $x$ be a non-zero of a normed space $X$. Prove that there exists $f$ a bounded linear functional such that $||f|| = 1$ and $f(x) = ||x||$.

(b) (10 points) Let $X$ be a reflexive space, show that for every $f \in X'$, $||f|| = 1$, there exists $x \in X$ such that $||x|| = 1$ and $f(x) = 1$.

Hint: Use part (a).
9. (a) (10 points) Let \( \{\phi_n\} \) be an orthonormal basis in \( \mathcal{H} \). Let \( \{\alpha_n\} \) be a bounded sequence of real numbers. Show that

\[
A(\phi_n) = \alpha_n \phi_n
\]

defines a bounded linear operator \( A : \mathcal{H} \to \mathcal{H} \) such that

\[
||A|| = \sup_{n \in \mathbb{N}} |\alpha_n|
\]
10. (a) (10 points) Let $U$ and $V$ be normed spaces and let $A : U \to V$ be a linear operator. 
Show that the inverse operator $A^{-1} : \mathcal{R}(A) \to U$ exists and continuous if and only if $A$ is bounded below (i.e., there exists $c > 0$ such that $||Au|| \geq c||u||$ for all $u \in U$.)