King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 535
Functional Analysis

20 December 2018

Name: ____

This exam contains 11 pages (including this cover page) and 10 questions. Total of points is 130.

Distribution of Marks

Question	Points	Score
1	15	
2	18	
3	15	
4	10	
5	7	
6	15	
7	10	
8	20	
9	10	
10	10	
Total:	130	

1. (a) (10 points) Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$. Show that the following are equivalent

   1. $u \in \mathcal{M}^\perp$.
   2. $||u - v|| \geq ||u||$ for all $v \in \mathcal{M}$.

   (b) (5 points) Let $\mathcal{H}$ be an **inner product space**. For a non-zero vector $a \in \mathcal{H}$. Set

$$\mathcal{M} = \{u \in \mathcal{H} : (u, a) = 0\}.$$

   Find $\mathcal{M}^\perp$.

2. (a) (10 points) Let $\{\phi_1, \ldots, \phi_n\}$ be an orthonormal set of vectors in a Hilbert space $\mathcal{H}$ and $\mathcal{M} = \langle \phi_1, \ldots, \phi_n \rangle$. Let $x \in \mathcal{H}$ and set $y = \sum_{i=1}^{n} (x, \phi_i)\phi_i$. Show that

   1. y is orthogonal to $x - y$.
   2. $x - y \in \mathcal{M}^{\perp}$
   3. $d(x, \mathcal{M}) = ||x - y||$

   (b) (8 points) Let $\mathcal{M}$ be a one-dimensional subspace of $\mathcal{H}$. Let a be a non-zero element of $\mathcal{M}$. Show that

$$d(x, \mathcal{M}^{\perp}) = \frac{|\langle x, a \rangle|}{||a||}$$

3. (a) (15 points) Let $\mathcal{H}$ be an inner product space and let $p : \mathcal{H} \to \mathcal{H}$ be a projection

   1. Show that $\mathcal{H} = \mathcal{R}(p) \oplus \operatorname{Ker} p$.
   2. Recall that p is called an **orthogonal projection** if its null-space and its range are orthogonal.

      Show that if p is an orthogonal projection, then

      $$\operatorname{Ker} p = \mathcal{R}(p)^{\perp} \text{ and } \mathcal{R}(p) = (\operatorname{Ker} p)^{\perp}.$$

   3. Let $p : \mathcal{H} \to \mathcal{H}$ be a non-zero orthogonal projection. Show that p is continuous and $||p|| = 1$.

4. (a) (10 points) Let $\mathcal{M}$ be a non-empty, closed subspace of a Hilbert space $\mathcal{H}$.

    1. Show that there exists a unique projection $p_{\mathcal{M}} : \mathcal{H} \to \mathcal{H}$ such that $\mathcal{R}(p_{\mathcal{M}}) = \mathcal{M}$.
    2. Show that $d(u, \mathcal{M}) = ||u - p_{\mathcal{M}}(u)||$.

5. (a) (7 points) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Prove that for any $f \in \mathcal{M}'$, the functional $F : \mathcal{H} \to \mathbb{R}$ defined by

$$F(u) = f(pr_{\mathcal{M}}(u))$$

is an extension of f with $||F|| = ||f||$.

6. (a) (10 points) Let f be a continuous linear functional on a Hilbert space $\mathcal{H}$. Prove that there exists a unique $z \in \mathcal{H}$ such that $f(x) = \langle x, z \rangle$ and $||f|| = ||z||$.

   (b) (5 points) Let $\{\phi_n\}$ be an orthonormal basis of $\mathcal{H}$, show that

$$z = \sum_{n=1}^{\infty} \overline{f(\phi_n)}\, \phi_n.$$

7. (a) (10 points) Let $\mathcal{H}$ be a Hilbert space and $\{\phi_n\}$ be an orthonormal system. Prove that ϕ_n converges weakly to zero, that is, $\langle y, \phi_n \rangle \to 0$ for all $y \in \mathcal{H}$.

8. (a) (10 points) Let x be a non-zero of a normed space X. Prove that there exists f a bounded linear functional such that $||f|| = 1$ and $f(x) = ||x||$.

   (b) (10 points) Let X be a reflexive space, show that for every $f \in X'$, $||f|| = 1$, there exists $x \in X$ such that $||x|| = 1$ and $f(x) = 1$.
   Hint: Use part(a).

9. (a) (10 points) Let $\{\phi_n\}$ be an orthonormal basis in $\mathcal{H}$. Let $\{\alpha_n\}$ be a bounded sequence of real numbers. Show that

$$A(\phi_n) = \alpha_n \phi_n$$

defines a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ such that

$$||A|| = \sup_{n \in \mathbb{N}} |\alpha_n|$$

10. (a) (10 points) Let U and V be normed spaces and let $A : U \to V$ be a linear operator. Show that the inverse operator $A^{-1} : \mathcal{R}(A) \to U$ exists and continuous if and only if A is bounded below (i.e., there exists $c > 0$ such that $||Au|| \geq c||u||$ for all $u \in U$.)