**MATH 102 – Quiz 2** (Term 182) **Section** 13

**Exercise 1** (5 points)
Evaluate the following integrals:

$$\text{a) (2,5 pts)} \int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta} d\theta$$

$$\text{b) (2,5 pts)} \int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt{x}} dx$$

**Solution**

a) (2,5 pts) $\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta} d\theta = \int_0^{\pi/4} \left(1+\frac{1}{\cos^2\theta}\right) d\theta = [\theta + \tan(\theta)]_0^{\pi/4} = \frac{\pi}{4}+1$

b) (2,5 pts)

$$\begin{aligned}\int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt{x}} dx &= \int_1^{64} \left(x^{-1/2}+x^{1/3-1/2}\right) dx = \int_1^{64} \left(x^{-1/2}+x^{-1/6}\right) dx \\ &= \left[\frac{x^{-1/2+1}}{-1/2+1}+\frac{x^{-1/6+1}}{-1/6+1}\right]_1^{64} \\ &= \left[2x^{1/2}+\frac{6}{5}x^{5/6}\right]_1^{64} = 51+\frac{1}{5}\end{aligned}$$

**Exercise 2 (5 points)**

a) If $f$ is a continous function, prove that: $\int\limits_0^{\pi/2} f(\cos(x))dx = \int\limits_0^{\pi/2} f(\sin(x))dx$

b) Use part a) to evaluate $\int\limits_0^{\pi/2} \cos^2(x)dx$ and $\int\limits_0^{\pi/2} \sin^2(x)dx$.

**Solution**

a) (2,5 pts) The integral is of the form $\int\limits_0^{\pi/2} f(\cos(x))dx = \int\limits_0^{\pi/2} f(g(x))dx$ where $g(x) = \cos(x)$. We know that $g'(x) = -\sin(x) = -\sqrt{1-\cos^2(x)} = -\sqrt{1-g(x)^2}$.

Therefore $\int\limits_0^{\pi/2} f(\cos(x))dx = \int\limits_0^{\pi/2} f(\cos(x))\frac{-\sin(x)}{-\sin(x)}dx = \int\limits_0^{\pi/2} -\frac{f(g(x)).g'(x)}{\sqrt{1-g(x)^2}}dx$

If we put $y = g(x)$, then by the substitution formula for definite integrals we have

$\int\limits_0^{\pi/2} -\frac{f(g(x)).g'(x)}{\sqrt{1-g(x)^2}}dx = \int\limits_{g(\pi/2)}^{g(0)} \frac{f(y)}{\sqrt{1-y^2}}dy = \int\limits_0^{1} \frac{f(y)}{\sqrt{1-y^2}}dy$ (1)

The same thing may be performed for the integral $\int\limits_0^{\pi/2} f(\sin(x))dx$ with $g(x) = \sin(x)$ and $g'(x) = \cos(x)$

We find $\int\limits_0^{\pi/2} f(\sin(x))dx = \int\limits_0^{1} \frac{f(y)}{\sqrt{1-y^2}}dy$ (2). From (1) and (2) it is immediate that $\int\limits_0^{\pi/2} f(\sin(x))dx$ $= \int\limits_0^{\pi/2} f(\cos(x))dx$.

b) (2,5 pts) Let us notice that $\int\limits_0^{\pi/2} \cos^2(x))dx = \int\limits_0^{\pi/2} f(\cos(x))dx$

and $\int\limits_0^{\pi/2} \sin^2(x))dx = \int\limits_0^{\pi/2} f(\sin(x))dx$ with $f(x) = x^2$.

By using part (a) it is obvious that $\int\limits_0^{\pi/2} \cos^2(x))dx = \int\limits_0^{\pi/2} \sin^2(x))dx$

But $\int\limits_0^{\pi/2} \cos^2(x))dx + \int\limits_0^{\pi/2} \sin^2(x))dx = \int\limits_0^{\pi/2} 1dx = \frac{\pi}{2}$

Then $\int\limits_0^{\pi/2} \cos^2(x))dx = \int\limits_0^{\pi/2} \sin^2(x))dx = \frac{\pi}{4}$