Exercise 1 (5 points)
Evaluate the following integrals:

\[
\begin{align*}
\text{a) (2,5 pts)} & \int_0^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta} \, d\theta \\
\text{b) (2,5 pts)} & \int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}} \, dx
\end{align*}
\]

Solution

a) (2,5 pts)
\[
\int_0^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta} \, d\theta = \int_0^{\pi/4} \left(1 + \frac{1}{\cos^2 \theta}\right) \, d\theta = [\theta + \tan(\theta)]_0^{\pi/4} = \frac{\pi}{4} + 1
\]

b) (2,5 pts)
\[
\int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}} \, dx = \int_1^{64} (x^{-1/2} + x^{1/3-1/2}) \, dx = \int_1^{64} (x^{-1/2} + x^{-1/6}) \, dx
\]
\[
= \left[ \frac{x^{-1/2+1}}{-1/2 + 1} + \frac{x^{-1/6+1}}{-1/6 + 1} \right]_1^{64} = \left[ 2x^{1/2} + \frac{6}{5}x^{5/6} \right]_1^{64} = 51 + \frac{1}{5}
\]
Exercise 2 (5 points)

a) If \( f \) is a continuous function, prove that:
\[
\int_0^{\pi/2} f(\cos(x)) \, dx = \int_0^{\pi/2} f(\sin(x)) \, dx
\]

b) Use part a) to evaluate \( \int_0^{\pi/2} \cos^2(x) \, dx \) and \( \int_0^{\pi/2} \sin^2(x) \, dx \).

Solution

a) (2,5 pts) The integral is of the form
\[
\int_0^{\pi/2} f(\cos(x)) \, dx = \int_0^{\pi/2} f(g(x)) \, dx
\]
where \( g(x) = \cos(x) \).

We know that \( g'(x) = -\sin(x) = -\sqrt{1 - \cos^2(x)} = -\sqrt{1 - g(x)^2} \).

Therefore
\[
\int_0^{\pi/2} f(\cos(x)) \, dx = \int_0^{\pi/2} f(\cos(x)) \frac{-\sin(x)}{-\sin(x)} \, dx = \int_0^{\pi/2} -\frac{f(g(x)) \cdot g'(x)}{\sqrt{1 - g(x)^2}} \, dx
\]

If we put \( y = g(x) \), then by the substitution formula for definite integrals we have
\[
\int_0^{\pi/2} -\frac{f(g(x)) \cdot g'(x)}{\sqrt{1 - g(x)^2}} \, dx = \int_0^{g(0)} \frac{f(y)}{\sqrt{1 - y^2}} \, dy \quad (1)
\]

The same thing may be performed for the integral \( \int_0^{\pi/2} f(\sin(x)) \, dx \) with \( g(x) = \sin(x) \) and \( g'(x) = \cos(x) \).

We find
\[
\int_0^{\pi/2} f(\sin(x)) \, dx = \int_0^{g(0)} \frac{f(y)}{\sqrt{1 - y^2}} \, dy \quad (2).
\]

From (1) and (2) it is immediate that
\[
\int_0^{\pi/2} f(\sin(x)) \, dx = \int_0^{\pi/2} f(\cos(x)) \, dx = \int_0^{\pi/2} \frac{1}{\sqrt{1 - y^2}} \, dy
\]

b) (2,5 pts) Let us notice that
\[
\int_0^{\pi/2} \cos^2(x) \, dx = \int_0^{\pi/2} f(\cos(x)) \, dx
\]
and
\[
\int_0^{\pi/2} \sin^2(x) \, dx = \int_0^{\pi/2} f(\sin(x)) \, dx
\]
with \( f(x) = x^2 \).

By using part (a) it is obvious that
\[
\int_0^{\pi/2} \cos^2(x) \, dx = \int_0^{\pi/2} \sin^2(x) \, dx
\]

But
\[
\int_0^{\pi/2} \cos^2(x) \, dx + \int_0^{\pi/2} \sin^2(x) \, dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}
\]

Then
\[
\int_0^{\pi/2} \cos^2(x) \, dx = \int_0^{\pi/2} \sin^2(x) \, dx = \frac{\pi}{4}
\]