Exercise 1 (5 points)
Let $A$ be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

a) Using right endpoints find an expression of $A$ as a limit. Do not evaluate the limit.
b) Estimate the area by taking the sample points to be midpoints and using four subintervals.

Solution
a) (2.5 pts) The expression for the area $A$ of a function is given by

$$A = \lim_{n \to +\infty} \sum_{i=1}^{i=n} f(x_i) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = x_0 + i\Delta x$. In our case $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$

Then $A = \lim_{n \to +\infty} \sum_{i=1}^{i=n} e^{-\frac{2i}{n}} \frac{2}{n}$

b) (2.5 pts) The midpoints of the four intervals are $x_1^* = \frac{1}{4}, x_2^* = \frac{3}{4}, x_3^* = \frac{5}{4}$ and $x_4^* = \frac{7}{4}$

Then an estimation of the area is given by:

$$A = \frac{1}{2} \left( e^{-\frac{1}{4}} + e^{-\frac{3}{4}} + e^{-\frac{5}{4}} + e^{-\frac{7}{4}} \right)$$
Exercice 2 (5 points)

1) Express the limit as an integral on the given interval \( \sum_{i=1}^{n} x_i \ln(1 + x_i^2) \Delta x \), on \([2, 6]\)

2) Use the definition of the integral as limit of Riemann sums to evaluate the integral \( \int_{-1}^{5} (1+3x)dx \)

Solution

1) (2,5 pts) It is clear that the limit under consideration is of the form

\[
A = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i) \Delta x = \int_{a}^{b} f(x)dx
\]

then

\[
\lim_{n \to +\infty} \sum_{i=1}^{n} x_i \ln(1 + x_i^2) \Delta x = \int_{2}^{6} x \ln(1 + x^2)dx
\]

2) (2,5 pts) We know that \( \int_{-1}^{5} (1 + 3x)dx \) is the limit of Riemann sums that is

\[
\int_{-1}^{5} (1 + 3x)dx = \lim_{n \to +\infty} \sum_{i=1}^{n} (1 + 3x_i) \Delta x, \text{ where } \Delta x = \frac{5 - (-1)}{n} = \frac{6}{n} \text{ and } x_i = -1 + \frac{i}{n}
\]

\[
\sum_{i=1}^{n} (1 + 3x_i) \Delta x = \frac{6}{n} \sum_{i=1}^{n} (1 + 3 \left(-1 + \frac{i}{n}\right)) = \frac{6}{n} \sum_{i=1}^{n} \left(-2 + i \frac{18}{n}\right) = \frac{6}{n} \left[-2n + \frac{18 n(n + 1)}{2}\right]
\]

Then

\[
\int_{-1}^{5} (1 + 3x)dx = \lim_{n \to +\infty} \frac{6}{n} \left[-2n + \frac{18 n(n + 1)}{2}\right] = -12 + 54 = 42,
\]

because \( \lim_{n \to +\infty} \frac{n(n + 1)}{n^2} = 1. \)