Exercise 1 (5 points)
Evaluate the following integrals:

a) (2.5 pts) \[ \int_0^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta} d\theta \]

b) (2.5 pts) \[ \int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}} dx \]

Solution

a) (2.5 pts) \[ \int_0^{\pi/4} \frac{1 + \cos^2 \theta}{\cos^2 \theta} d\theta = \left[ \theta + \tan(\theta) \right]_0^{\pi/4} = \frac{\pi}{4} + 1 \]

b) (2.5 pts) \[ \int_1^{64} \frac{1 + \sqrt[3]{x}}{\sqrt{x}} dx = \int_1^{64} \left( x^{-1/2} + x^{1/3-1/2} \right) dx = \int_1^{64} \left( x^{-1/2} + x^{-1/6} \right) dx \]
\[ = \left[ \frac{x^{-1/2+1}}{-1/2 + 1} + \frac{x^{-1/6+1}}{-1/6 + 1} \right]_1^{64} \]
\[ = \left[ 2x^{1/2} + \frac{6}{5}x^{5/6} \right]_1^{64} = 51 + \frac{1}{5} \]
Exercise 2 (5 points)

a) If \( f \) is a continuous function, prove that:
\[
\frac{\pi}{2} \int_0^0 f(\cos(x))dx = \frac{\pi}{2} \int_0^0 f(\sin(x))dx
\]

b) Use part a) to evaluate \( \int_0^0 \cos^2(x)dx \) and \( \int_0^0 \sin^2(x)dx \).

Solution

a) (2.5 pts) The integral is of the form
\[
\frac{\pi}{2} \int_0^0 f(\cos(x))dx = \frac{\pi}{2} \int_0^0 f(g(x))dx
\]
where \( g(x) = \cos(x) \).

We know that \( g'(x) = -\sin(x) = -\sqrt{1 - \cos^2(x)} = -\sqrt{1 - g(x)^2} \).

Therefore
\[
\frac{\pi}{2} \int_0^0 f(\cos(x))dx = \frac{\pi}{2} \int_0^0 f(\cos(x)) \frac{-\sin(x)}{-\sin(x)}dx = \frac{\pi}{2} \int_0^0 \frac{f(g(x)) \cdot g'(x)}{\sqrt{1 - g(x)^2}}dx
\]

If we put \( y = g(x) \), then by the substitution formula for definite integrals we have
\[
\frac{\pi}{2} \int_0^0 -\frac{f(g(x)) \cdot g'(x)}{\sqrt{1 - g(x)^2}}dx = \frac{\pi}{2} \int_0^0 \frac{f(y)}{\sqrt{1 - y^2}}dy \tag{1}
\]

The same thing may be performed for the integral \( \int_0^0 f(\sin(x))dx \) with \( g(x) = \sin(x) \) and \( g'(x) = \cos(x) \).

We find
\[
\frac{\pi}{2} \int_0^0 f(\sin(x))dx = \frac{\pi}{2} \int_0^0 \frac{f(y)}{\sqrt{1 - y^2}}dy \tag{2}
\]
From (1) and (2) it is immediate that
\[
\int_0^0 f(\sin(x))dx = \int_0^0 f(\cos(x))dx
\]

b) (2.5 pts) Let us notice that
\[
\frac{\pi}{2} \int_0^0 \cos^2(x)dx = \frac{\pi}{2} \int_0^0 f(\cos(x))dx
\]

and
\[
\frac{\pi}{2} \int_0^0 \sin^2(x)dx = \frac{\pi}{2} \int_0^0 f(\sin(x))dx \text{ with } f(x) = x^2.
\]

By using part (a) it is obvious that
\[
\frac{\pi}{2} \int_0^0 \cos^2(x)dx = \frac{\pi}{2} \int_0^0 \sin^2(x)dx
\]

But
\[
\frac{\pi}{2} \int_0^0 \cos^2(x)dx + \frac{\pi}{2} \int_0^0 \sin^2(x)dx = \frac{\pi}{2} \int_0^0 1dx = \frac{\pi}{2}
\]

Then
\[
\frac{\pi}{2} \int_0^0 \cos^2(x)dx = \frac{\pi}{4}
\]