Exercise 1 (5 points)
Which of the following sequences converge and which diverge? Find the limit of each convergent sequence.

a) \( a_n = 2 + (0, 1)^n \)

Solution

a) It is clear that \( 0, 1 < 1 \), then \( \lim_{n \to +\infty} (0, 1)^n = 0 \). Therefore \( \lim_{n \to +\infty} 2 + (0, 1)^n = 2 \).

b) \( b_n = \frac{n + (-1)^n}{n} = 1 + \frac{(-1)^n}{n} \) and \( \frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \)

It is obvious that the sequences \( \left( \frac{-1}{n} \right) \) and \( \left( \frac{1}{n} \right) \) converge to 0. Therefore the sequence \( \left( \frac{(-1)^n}{n} \right) \) converges to 0 and we conclude that the sequence \( (b_n) \) converges to 1.

c) \( \lim_{n \to +\infty} \frac{n^2 - 2n + 1}{n - 1} = \lim_{n \to +\infty} \frac{n^2}{n} = \lim_{n \to +\infty} n = +\infty \), then \((c_n)\) is divergent.

d) We know that \( \frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \) and that \( \left( \frac{-1}{n} \right) \) and \( \left( \frac{1}{n} \right) \) converge to 0, then \( \left( \frac{\sin(n)}{n} \right) \) converges to 0.
Exercise 2 (5 points)

Consider the following series \( \sum_{n=1}^{+\infty} 2^{2n}3^{1-n} \).

1) Write this series as a geometric series. Is this series convergent or divergent?

2) Show that the series \( \sum_{n=1}^{+\infty} \frac{1}{n(n+1)} \) is convergent and evaluate its sum.

Solution

1) We can write \( 2^{2n}3^{1-n} = \left(\frac{2^2}{3}\right)^n = \frac{4}{3}^n \)

Then the series in question may be written as a geometric series as follows

\[
\sum_{n=1}^{+\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{+\infty} \frac{4}{3}^n
\]

with ratio \( \frac{4}{3} > 1 \). Then this series is divergent.

2) Let \( S_n \) the partial sum of order \( n \), \( S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} \)

Notice that we can write \( \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \). Then

\[
S_n = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) - \left( \frac{1}{2} + \ldots + \frac{1}{n+1} \right) = \left( 1 - \frac{1}{n+1} \right)
\]

Then \( \lim_{n \to +\infty} S_n = \lim_{n \to +\infty} \left( 1 - \frac{1}{n+1} \right) = 1 \).

Therefore the series is convergent and its sum is equal to 1.