Exercise 1
Let $n$ be a positive integer with decimal representation $a_0 a_1 \ldots a_k$ ($k \geq 2$).
Then, in the ring $\mathbb{Z}_8$, $n$ is equal to:

(a) $a_0 - a_1 + \cdots + (-1)^k a_k$
(b) $a_0 - 2a_1 + 4a_2$
(c) $-4a_{k-2} + 2a_{k-1} + a_k$
(d) $a_0 + 2a_1 + 4a_2$
(e) $4a_{k-2} - 2a_{k-1} + a_k$

Exercise 2
The equation $x^{30} = 1 \pmod{43}$ has:

(a) 2 distinct solutions
(b) 5 distinct solutions
(c) 6 distinct solutions
(d) 11 distinct solutions
(e) 30 distinct solutions

Exercise 3
Let $p$ be an arbitrary prime number. Then:

(a) The number of all monic non-irreducible quadratics in $\mathbb{Z}_p[X]$ is equal to $\frac{p(p-1)}{2}$
(b) The number of all monic irreducible quadratics in $\mathbb{Z}_p[X]$ is equal to $\frac{p(p+1)}{2}$
(c) $X^2 + X + 1$ is the only monic irreducible quadratic in $\mathbb{Z}_p[X]$
(d) The number of all monic non-irreducible quadratics in $\mathbb{Z}_p[X]$ is equal to $\frac{p(p+1)}{2}$
(e) The number of all monic irreducible quadratics in $\mathbb{Z}_p[X]$ is equal to $p^2$
Exercise 4
Let $G := \{1,4,11,14,16,19,26,29,31,34,41,44\}$ be a group under multiplication modulo 45. Then:

(a) $G = \langle 4 \rangle \oplus \langle 19 \rangle$
(b) $G = \langle 4 \rangle \oplus \langle 26 \rangle$
(c) $G = \langle 11 \rangle \oplus \langle 26 \rangle$
(d) $G = \langle 11 \rangle \oplus \langle 31 \rangle$
(e) $G = \langle 16 \rangle \oplus \langle 19 \rangle$

Exercise 5
The lattice of all subgroups of $\mathbb{Z}_{66}$ is:

(a) (b) (c) (d) (e)

Exercise 6
Let $n, m$ be two integers such that $1 \leq n \leq m$. Let $G$ be a finite group of order $m$ and let $a$ be an element of $G$ of order $n$ such that $gag^{-1} \in \langle a \rangle \ \forall g \in G$. Assume that $(\phi(n), m) = 1$, where $\phi$ denotes the Euler function. Then:

(a) $\langle a \rangle = G$
(b) $G/Z(G)$ is not cyclic
(c) $|Z(G)| = \phi(n)$
(d) $\langle a \rangle \subseteq Z(G)$
(e) $|G/Z(G)| = n$
Exercise 7
When $x^{48}$ is divided by $x + 10$ in $\mathbb{Z}_{13}$, then the remainder is:

(a) 1  
(b) 2  
(c) 3  
(d) 4  
(e) 5

Exercise 8
Let $A_8$ denote the alternating group of degree 8 and let $\sigma \in A_8$ with $\sigma = \alpha_1 \alpha_2 ... \alpha_k$ where the $\alpha_i$'s are non-trivial disjoint $r_i$-cycles ($r_i \geq 2$). Then, all possible values for $k$ are:

(a) 1, 2, 3  
(b) 2, 3, 4  
(c) 2, 3  
(d) 3, 4  
(e) 2, 4

Exercise 9
Let $S$ be the group of permutations of a finite set and let $\alpha$ and $\beta$ be any two distinct transpositions in $S$. Then:

(a) $|\alpha \beta| = 1$  
(b) $|\alpha \beta| = 2$  
(c) $|\alpha \beta| = 1$ or 2  
(d) $|\alpha \beta| = 2$ or 3  
(e) $|\alpha \beta| = 1$ or 3

Exercise 10
Let $A_{10}$ denote the alternating group of degree 10 and let $\sigma \in A_{10}$ with $\sigma = \alpha_1 \alpha_2 ... \alpha_k$ where the $\alpha_i$'s are non-trivial disjoint $r_i$-cycles ($r_i \geq 2$). Then, the maximum order $\sigma$ can have is:

(a) 10  
(b) 15  
(c) 21  
(d) 30  
(e) 42
Exercise 11
Let $G$ be a group such that $x^2 = 1$, for each $x \in G$. Then:
(a) $|G| = 2$
(b) $G$ is abelian
(c) $G$ is cyclic
(d) $|x| = 2$, for each $x \in G$
(e) $Z(G) = \{1\}$

Exercise 12
Let $G := \{1,8,12,14,18,21,27,31,34,38,44,47,51,53,57,64\}$ be a group under multiplication modulo 65. We know that $G$ has three elements of order 2 and all the other elements (except 1) have order equal to 4. Then $G$ is isomorphic to:
(a) $\mathbb{Z}_{16}$
(b) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
(c) $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
(d) $\mathbb{Z}_4 \oplus \mathbb{Z}_4$
(e) $\mathbb{Z}_8 \oplus \mathbb{Z}_2$

Exercise 13
If $x$ is an element of a cyclic group of order 105, and exactly two of $x^4, x^6, x^{32}$ are equal, then $|x^{15}|$ is equal to:
(a) 15
(b) 7
(c) 5
(d) 3
(e) 1

Exercise 14
Let $G$ be a group. Which of the following statement(s) is/are true:
I. If $G$ is noncyclic, then there exists a proper non-cyclic subgroup of $G$.
II. If $a, b \in G$ and $|a|$ and $|b|$ are finite, then $|ab|$ is finite.
III. $\bigcap_{a \in G} c(a) = G$ if and only if $G$ is abelian.
(a) I and II only
(b) II and III only
(c) III only
(d) II only
(e) I and III only
Exercise 15
Let \( p \) be a prime \( \geq 3 \) and consider the group \( G = \mathbb{Z}_p \oplus \mathbb{Z}_p \). Then, the number of subgroups of \( G \) of order \( p \) is equal to:

(a) 1
(b) \( p - 1 \)
(c) \( p \)
(d) \( p + 1 \)
(e) \( p^2 - 1 \)

Exercise 16
Let \( G \) be a group of order 231 and let \( a, b \) be nonidentity elements of \( G \) with \( |a| \neq |b| \). Let \( H \) be a subgroup of \( G \) containing both \( a \) and \( b \). Which one of the following numbers is \textbf{NOT} a possible order of \( H \):

(a) 33
(b) 77
(c) 11
(d) 21
(e) All the above numbers are possible orders of \( H \)

Exercise 17
Up to isomorphism, the number of additive abelian groups of order 16 that have the property \( x + x + x + x = 0 \) for all \( x \) in the group is equal to:

(a) 2
(b) 5
(c) 1
(d) 4
(e) 3

Exercise 18
Let \( G \) be a group of order 77. Which of the following numbers are possible orders of \( Z(G) \):

(a) \{1,77\}
(b) \{1,7,77\}
(c) \{1,11,77\}
(d) \{7,11,77\}
(e) \{1,7,11,77\}
Exercise 19
If $F$ is a field of order 169, then the characteristic of $F$ is equal to:

(a) 1
(b) 13
(c) 0
(d) 3
(e) 169

Exercise 20
Consider the ring $R = \mathbb{Z}[i]$ and the ideal $I := (1 - i)$ in $R$. Then:

(a) $\frac{R}{I} = \{0\}$
(b) $\frac{R}{I}$ is a ring with zero-divisors
(c) $\frac{R}{I} = \mathbb{Z}$
(d) $\frac{R}{I}$ is a field
(e) None of the above statements is correct

Exercise 21
Which one is NOT a field:

(a) $\frac{\mathbb{Z}_3[X]}{(X+1)}$
(b) $\frac{\mathbb{Z}_5[X]}{(X^5+1)}$
(c) $\frac{\mathbb{Z}_3[X]}{(X^4+2X+2)}$
(d) $\frac{\mathbb{Z}_3[X]}{(X^3+2X+1)}$
(e) All are fields

Exercise 22
For a ring $R$, let $Aut(R)$ denote the set of all ring isomorphisms from $R$ to $R$, and recall that a ring homomorphism, by our definition, must carry 1 to 1. Which one of the following statements is true:

(a) $Aut(\mathbb{Q}) = \emptyset$
(b) $|Aut(\mathbb{Z})| \geq 2$
(c) $|Aut(\mathbb{Q})| = |Aut(\mathbb{R})|$
(d) $Aut(\mathbb{Z}) = \emptyset$
(e) $|Aut(\mathbb{R})| = \infty$
Exercise 23
The ring $\mathbb{R}[X]/(X^4-1)$ is isomorphic to:

(a) $\mathbb{R}$
(b) $\mathbb{R} \times \mathbb{R}$
(c) $\mathbb{R} \times \mathbb{C}$
(d) $\mathbb{R} \times \mathbb{R} \times \mathbb{C}$
(e) $\mathbb{R} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}$

Exercise 24
The polynomial $f: = X^4 + 1$ is NOT irreducible over

(a) $\mathbb{Q}$
(b) $\mathbb{Q}[i]$
(c) $\mathbb{Q}[\sqrt{2}]$
(d) $\mathbb{Q}[\sqrt{3}]$
(e) $f$ is irreducible over all the above fields

Exercise 25
Let $R$ be a commutative ring with the property:
Every decreasing chain of ideals $I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$ must be finite in length
(i.e., there is a positive integer $n$ such that $I_n = I_{n+1}$). Then:

(a) $R$ is a field
(b) $R$ is a UFD
(c) $R$ has only one maximal ideal
(d) Every prime ideal of $R$ is maximal
(e) Such a ring $R$ does NOT exist