King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 435 Final Exam
The Second Semester of 2018-2019 (182)
Time Allowed: 120mn

Name: ID number:

Textbooks are not authorized in this exam

Problem #	Marks	Maximum Marks
1		20
2		20
3		20
4		20
5		20
Total		100

Problem 1: Consider the system

$$\frac{dx}{dt} = x(2-2x-y) \quad , \quad x>0 \ , \ y>0.$$
$$\frac{dy}{dt} = y(2-x-2y).$$

1.)(5pts) Find the critical solution. $(x_0, y_0) \neq (0,0)$
2.)(8pts) Show that the system is almost linear in the neighborhood of the critical point. (x_0, y_0).
3.)(7pts) Study the stability of the critical point. (x_0, y_0).

Solution

1.) $\begin{cases} x(2-2x-y)=0 \\ y(2-x-2y)=0 \end{cases}$ $\quad x\neq 0, \ y \neq 0 \Rightarrow \begin{cases} 2-2x-y=0 \\ 2-x-2y=0 \end{cases} \Rightarrow y = 2-2x$

$\Rightarrow 2-x-2(2-2x)=0$

$\Rightarrow x = \frac{2}{3}$ $\quad A\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$

2.) $u = x - \frac{2}{3}$
$y = v - \frac{2}{3}$ $\Rightarrow \begin{cases} \frac{du}{dt} = -\frac{2}{3}(2u+v) - u(2u+v) \\ \frac{dv}{dt} = -\frac{2}{3}(u+2v) - v(u+2v) \end{cases}$

Let $F(u,v) = \begin{pmatrix} -u(2u+v) \\ -v(u+2v) \end{pmatrix}$; $X' = AX + F$

$$|F(u,v)| = |u|\,|2u+v| + |v|\,|u+2v|$$
$$\leq |u|\, 2(|u|+|v|) + |v|\, 2(|u|+|v|) \leq 2(|u|+|v|)^2$$

$$\Rightarrow \lim_{|u|+|v|\to 0} \frac{|F(u,v)|}{|u|+|v|} = 0$$

$\Rightarrow$ The system is almost linear

3.) $\begin{vmatrix} -\frac{4}{3}-\lambda & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3}-\lambda \end{vmatrix} = 0 \Leftrightarrow \left(-\frac{4}{3}-\lambda\right)^2 - \frac{4}{9} = 0, \ \left(\frac{4}{3}+\lambda\right)^2 = \frac{4}{9}$

$\lambda = -\frac{2}{3}, \ \lambda = -2$

The critical point is asymptotically stable.

Problem 2: Consider the ω- periodic linear system $X' = A(t)X$, where

$$A(t) = \begin{pmatrix} 0 & 1 \\ a(t) & 0 \end{pmatrix}.$$

Let λ_j and ρ_j the Floquet multipliers and corresponding exponents of the system ($\lambda_j = e^{\omega \rho_j}$).
Study the stability of $X = 0$
1.)(10pts) if $\lambda_1 + \lambda_2 = -1$.
2.)(10pts) if $\lambda_1 + \lambda_2 = 4$.

Solution

1) $\lambda_1 \cdot \lambda_2 = e^{\int_0^\omega Tr(A)\,ds}$; $Tr(A) = 0 \Rightarrow \lambda_1 \cdot \lambda_2 = 1$

We solve the system $\begin{cases} \lambda_1 \cdot \lambda_2 = 1 \\ \lambda_1 + \lambda_2 = -1 \end{cases}$ $\Rightarrow \lambda_2 = \frac{1}{\lambda_1}$ $\Rightarrow \lambda_1 + \frac{1}{\lambda_1} = -1$

$\lambda_1^2 + \lambda_1 + 1 = 0$, $\lambda_1 = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$

$\lambda_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$\lambda_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$|\lambda_1| = 1$. and $|\lambda_2| = 1$

$\Rightarrow X = 0$ is stable

2) $\begin{cases} \lambda_1 \lambda_2 = 1 \\ \lambda_1 + \lambda_2 = 4 \end{cases}$, $\lambda_1 + \frac{1}{\lambda_1} = 4$, $\lambda_1^2 - 4\lambda_1 + 1 = 0$

$\lambda_1 = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\lambda_1 = 2 - \sqrt{3} \Rightarrow \lambda_2 = 2 + \sqrt{3}$
$\lambda_1 = 2 + \sqrt{3} \Rightarrow \lambda_2 = 2 - \sqrt{3}$

$|\lambda_2| > 1$

$\Rightarrow X = 0$ is unstable.

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Problem 3: Consider the system

$$\frac{dx}{dt} = x + xy^2$$
$$\frac{dy}{dt} = y + x^2y.$$

Let $V(x,y) = x^2 + xy + y^2$.
1.)(8pts) Compute the function $\frac{dV}{dt}$.
2.)(12pts) Deduce the stability of the critical point $(0,0)$.

Hint: A function $f(x,y)$ on region D
- negative definite on D if $\nabla f = 0$; $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} < 0$ and $f_{yy} < 0$ at the point $(0,0)$.
- positive definite on D if $\nabla f = 0$; $f_{xx}f_{yy} - f_{xy}^2 > 0$, $f_{xx} > 0$ and $f_{yy} > 0$ at the point $(0,0)$.

Solution

1) $\frac{dx}{dt} = x + xy^2 \quad \times x \Rightarrow \frac{1}{2}\frac{d}{dt}x^2 = x^2 + x^2y^2 \Rightarrow \frac{d}{dt}x^2 = 2x^2 + 2x^2y^2$ (1)

$\frac{dy}{dt} = y + x^2y \quad \times y \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 = y^2 + x^2y^2 \Rightarrow \frac{d}{dt}y^2 = 2y^2 + 2x^2y^2$ (2)

$\frac{d}{dt}(xy) = x\frac{dy}{dt} + y\frac{dx}{dt} = x(y + x^2y) + y(x + xy^2)$
$= 2xy + x^3y + xy^3$ (3)

(1) + (2) + (3) $\Rightarrow$ $\frac{d}{dt}V = \underbrace{2x^2 + 2y^2 + 2xy + 4x^2y^2 + yx^3 + xy^3}_{V^*(x,y)}$

2) $V^*_x = 4x + 2y + 8xy^2 + 3x^2y + y^3$
$V^*_y = 4y + 2x + 8x^2y + 3xy^2 + x^3$ $\Rightarrow \nabla V^*|_{(0,0)} = (0,0)$

$V^*_{xx} = 4 + 8y^2 + 6xy$
$V^*_{yy} = 4 + 8x^2 + 6xy$
$V^*_{xy} = 2 + 16xy + 3x^2 + 3y^2$

$V^*_{xx}V^*_{yy} - V^{*2}_{xy}\Big|_{(0,0)} = 4 \cdot 4 - 4 = 12 > 0$

$V^*_{xx}|_{(0,0)} = 4 > 0$, $V^*_{yy}(0,0) = 4 > 0$.

V^* is positive definite on D
V is positive definite on $\mathbb{R}^2$ $\Rightarrow$ $(0,0)$ is unstable

(3)

Problem 4:(20pts) We set $g(s) = s - s^3$. Assume $g^2(s) \le 4\int_0^s g(\sigma)d\sigma$, for all $|s| \le \frac{1}{4}$. Consider the system

$$\frac{dx}{dt} = y, \tag{1}$$
$$\frac{dy}{dt} = -y - g(x). \tag{2}$$

Find a domain D containing the point $(0,0)$, a positive definite function $V(x,y)$ such that $\frac{dV}{dt}$ is negative definite on D.

Solution

$$\frac{dy}{dt} = -y - g(x) \quad \times x'$$

$$\frac{1}{2}\frac{d}{dt}y^2 = -y^2 - \frac{d}{dt}\int_0^x g(s)ds \quad (\text{Since } x' = y)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{y^2}{2} + \int_0^x g(s)ds\right) = -y^2. \quad ①$$

$$\sigma\frac{d}{dt}(y g(x)) = \sigma y' g(x) + \sigma y x' g'(x)$$
$$= -\sigma y g(x) - \sigma g^2(x) + \sigma y^2 g'(x) \quad ②$$

① + ② $\Rightarrow$

$$\frac{d}{dt}\left(\frac{y^2}{2} + \int_0^x g(s)ds + \sigma y g(x)\right) = -y^2 - \sigma g^2(x) - \sigma y g(x) + \sigma y^2 g'(x)$$

$$V(x,y) = \frac{y^2}{2} + \int_0^x g(s)ds + \sigma y g(x)$$

$$V^*(x,y) = -y^2 - \sigma g^2(x) - \sigma y g(x) + \sigma y^2 g'(x)$$

$$|y g(x)| \le \frac{1}{2}y^2 + \frac{g^2(x)}{2} \le \frac{y^2}{2} + 2\int_0^x g(s)ds$$

$$\Rightarrow \sigma y g(x) \ge -\frac{\sigma}{2}y^2 - 2\sigma\int_0^x g(s)ds$$

$$\Rightarrow V(x,y) \ge \frac{1}{2}(1-\sigma)y^2 + (1-2\sigma)\int_0^x g(s)ds$$
$$\ge \frac{1}{2}\left(y^2 + \int_0^x g(s)ds\right), \text{ if } \sigma < 1/4.$$

Now, $|g'(x)| \le \max_{|x| \le \frac{1}{4}} |g'(x)| = M$

$$\Rightarrow -y^2 g'(x) \ge -My^2$$

$$-V^* = y^2 + \sigma g^2(x) + \underbrace{\sigma y g(x)}_{\ge \sigma\left(-\frac{y^2}{2} - \frac{g^2(x)}{2}\right)} - \sigma y^2 g'(x)$$

$$\Rightarrow -V^* \ge \left(1 - \frac{\sigma}{2} - \sigma M\right)y^2 + \sigma\left(1 - \frac{1}{2}\right)g^2(x), \text{ if } \sigma < \frac{1}{1+2M}$$

$-V^*$ is negative definite on

$$D = \{(x,y) \mid |x| \le \tfrac{1}{4},\ |y| < \infty\}$$

V is positive definite on D.

$\Rightarrow (0,0)$ is Asymptotically stable.

Problem 5:(20pts) Consider the system

$$\frac{dx}{dt} = -x + \frac{1}{2}y$$
$$\frac{dy}{dt} = -y + \frac{1}{2}x.$$

Show that $(0,0)$ is globally asymptotically stable.

Solution

$$\frac{dx}{dt} = -x + \frac{y}{2} \quad \times x \Rightarrow \frac{1}{2}\frac{d}{dt}x^2 = -x^2 + \frac{xy}{2}$$

$$\frac{dy}{dx} = -y + \frac{x}{2} \quad \times y \Rightarrow \frac{1}{2}\frac{d}{dt}y^2 = -y^2 + \frac{xy}{2}$$

$$\frac{1}{2}\frac{d}{dt}(x^2+y^2) = \underbrace{-x^2-y^2+xy}_{V^*}$$

$$-V^* = x^2+y^2-xy$$

$$|xy| < \frac{1}{2}x^2 + \frac{y^2}{2} \Rightarrow xy \geq -\frac{1}{2}x^2 - \frac{y^2}{2}$$

$$\Rightarrow -V^* > x^2+y^2 - \frac{1}{2}(x^2+y^2)$$

$$-V > \frac{1}{2}(x^2+y^2)$$

V and $-V^*$ are positive definite on $\mathbb{R}^2$

$\Rightarrow$ $(0,0)$ is globally asymptotically stable.