King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Term 182 - Math 551 - Midterm

Question 1. A ring is called *local* if it has a unique maximal ideal.

(a) Let A be a ring and $\mathfrak{m} \neq A$ be an ideal of A so that every element in $A \setminus \mathfrak{m}$ is a unit. Show that $\mathfrak{m}$ is the only maximal ideal of A.

(b) Let A be a commutative ring and $\mathfrak{m}$ be a maximal ideal of A so that every element $1 + \mathfrak{m}$ is a unit. Show that A is local.

Question 2. Let $(P_i)_{\{i\in\Delta\}}$ be a family of A-modules. Show that $\bigoplus_{i\in\Delta} P_i$ is projective if and only if P_i is projective for every $i \in \Delta$.

Question 3. Let A be a commutative ring.

(a) Show that $N = \{a \in A | a^n = 0 \text{ for some integer } n \geq 1\}$ is an ideal of A.

(b) Show that N is contained in every prime ideal of A.

(c) Let $a_0 \notin N$ and $\mathcal{F}$ be the family of ideals of A that intersect (a_0) at 0.

- (i) Show that $\mathcal{F}$ has a maximal element which is a prime ideal of A.
- (ii) Show that a_0 is not contained in the intersection of all prime ideals of A.

(d) Deduce that N is equal to the intersection of all prime ideals of A.

Question 4. Let A be a commutative ring and S be a multiplicative subset of A. $S^{-1}A$ denotes the localization of A by S. In case $S = A \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of A, the localization $S^{-1}A$ is denoted by $A_\mathfrak{p}$.

(Part I)

(a) Let M be an A-module. Define $S^{-1}M$ in a manner analogous to $S^{-1}A$ and show that it is an $S^{-1}A$-module.

(b) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, then show that the sequence $0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0$ is an exact sequence of $S^{-1}A$-modules.

(c) Let N be a submodule of the A-module M. Show that $S^{-1}M/S^{-1}N$ is isomorphic to $S^{-1}(M/N)$.

(Part II)

(d) If M is a non zero A-module, then show that there exists a maximal ideal $\mathfrak{m}$ of A so that $M_\mathfrak{m}$ is non zero.

(e) Let M be an A-module. Show that the followings are equivalent:

- (i) $M = 0$;
- (ii) $M_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ of A;
- (iii) $M_\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m}$ of A.

(f) Show that the followings are equivalent:

- (i) The sequence $0 \to M' \to M \to M'' \to 0$ is exact;
- (ii) The sequence $0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$ is exact for every prime ideal $\mathfrak{p}$ of A;
- (iii) The sequence $0 \to M'_\mathfrak{m} \to M_\mathfrak{m} \to M''_\mathfrak{m} \to 0$ is exact for every maximal ideal $\mathfrak{m}$ of A.