King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Term 182 - Math 551 - Final Exam

Question 1. Let $\mathcal{C}$ be a category and $f : X \to Y$ and $g : X \to Z$ be two maps in $\mathcal{C}$. The *pushout* of f and g is the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle v} \\ Y & \xrightarrow[u]{} & T \end{array}$$

satisfying the property that if there exists another commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle v'} \\ Y & \xrightarrow[u']{} & R \end{array}$$

then, there exists a unique map $t : T \to R$ so that $u' = u \circ t$ and $v' = v \circ t$. Show that in the category of commutative rings, the pushout is the tensor product.

Question 2. Let (A, μ) be a local commutative Noetherian ring and M be a finitely generated A-module. Using the fact;

"If $0 \to M \to N \to P \to 0$ is an exact sequence of A-modules where P is a flat A-module, then $0 \to M \otimes_A Q \to N \otimes_A Q \to P \otimes_A Q \to 0$ is also an exact sequence where Q is any A-module."

to show that M is free if and only if M is flat.

Question 3. Show that if M is a Noetherian A-module, then $A/\mathrm{Ann}(M)$ is a Noetherian ring.

Question 4. Let A be a commutative ring. Show that an A-homomorphism $f : M \to N$ is injective if and only if the induced map $f^* : \mathrm{Hom_A(N, Q)} \to \mathrm{Hom_A(M, Q)}$ is surjective for all injective A-modules Q.

Question 5.

(0) Show the followings:
  (i) $IJ \subseteq I \cap J$ for any two ideals I and J.
  (ii) Let $I_1, \ldots, I_n$ be ideals and P be a prime ideal. If $\cap_{i=1}^{n} I_i$ is contained in P, then I_i is contained in P for some $i \in \{1, \ldots, n\}$.

(1) Let V be a k-vector space. Show that the followings are equivalent:
  (i) V is finite dimensional;
  (ii) V is Artinian as k-module;
  (iii) V is Noetherian as k-module;

(2) Let A be a ring in which zero ideal is product of finitely many maximal ideals (not necessarily distinct). Then A is Noetherian if and only if A is Artinian.

(3) Let A be a commutative Artinian ring. Prove the followings:
  (i) All the prime ideals of A are maximal.
  (ii) A has finitely many maximal ideals.
  (iii) Every nonempty set of ideals of A has a minimal element.
  (iv) The Jacobson radical of A is nilpotent.
  (v) A is Noetherian.

(4) Give an example of a Noetherian ring which is not Artinian. Explain your answer.