Math201.01, Quizzes # 3 & 4, Term 191

Name:

Solutions

ID #: **Serial #:**

1. [5 points] Find and classify the critical points of

$$f(x, y) = 3xy^2 - x^2 - 6y^2.$$

• $f_x(x,y) = 3y^2 - 2x$; $f_y(x,y) = 6xy - 12y$

$\longrightarrow \begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3y^2 - 2x = 0 & —(1) \\ 6xy - 12y = 0 & —(2) \end{cases}$ 1

$(2) \Rightarrow 6y(x-2) = 0 \Rightarrow x = 2, \ y = 0$

• $x = 2 \overset{(1)}{\Longrightarrow} 3y^2 - 4 = 0 \Rightarrow y^2 = \frac{4}{3} \Rightarrow y = \pm \frac{2}{\sqrt{3}}$

$\boxed{(x,y) = (2, \frac{2}{\sqrt{3}}), (2, -\frac{2}{\sqrt{3}})}$ 1

• $y = 0 \overset{(1)}{\Longrightarrow} 0 - 2x = 0 \Rightarrow x = 0 \Rightarrow \boxed{(x,y) = (0,0)}$ 0.5

$\longrightarrow$ f_x & f_y exists at all pts (x,y).

• Test: $f_{xx}(x,y) = -2$, $f_{yy}(x,y) = 6x - 12$; $f_{xy}(x,y) = 6y$

$D(x,y) = f_{xx}(x,y) \, f_{yy}(x,y) - f_{xy}^2(x,y)$ 1

$= -2(6x-12) - 36y^2 = -12x - 36y^2 + 24$

$(0,0)$: $D(0,0) = 24 > 0$ & $f_{xx}(0,0) = -2 < 0 \Rightarrow f$ has a local max at $(0,0)$

$(2, \frac{2}{\sqrt{3}})$: $D(2, \frac{2}{\sqrt{3}}) = -24 - 48 + 24 = -48 < 0 \Rightarrow f$ has a saddle pt at $(2, \frac{2}{\sqrt{3}})$

$(2, \frac{-2}{\sqrt{3}})$: $D(2, \frac{-2}{\sqrt{3}}) = -24 - 48 + 24 = -48 < 0 \Rightarrow f$ has a saddle pt at $(2, \frac{-2}{\sqrt{3}})$.

1.5

2. [7 points] Find the extreme values of

$$f(x,y,z) = (x+y)z \quad = xz + yz$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Sol: $g(x,y,z) = x^2+y^2+z^2-1$.

Solve the system:

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g = 0 \end{cases} \Longrightarrow \begin{cases} z = \lambda \cdot 2x & \text{—(1)} \\ z = \lambda \cdot 2y & \text{—(2)} \\ x+y = \lambda \cdot 2z & \text{—(3)} \\ x^2+y^2+z^2 = 1 & \text{—(4)} \end{cases}$$

$(1)\ \&\ (2) \Rightarrow \lambda \cdot 2x = \lambda \cdot 2y \Rightarrow \lambda x = \lambda y \Rightarrow \lambda(x-y) = 0 \Rightarrow \underline{\lambda = 0 \text{ or } y = x}$

Case 1: $\lambda = 0 \xrightarrow{(1)} \underline{z = 0} \xrightarrow{(3)} x + y = 0 \Rightarrow \underline{y = -x}$

$\xrightarrow{(4)} x^2 + x^2 + 0 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$\boxed{\text{points}: (x,y,z) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)} \xrightarrow{f} 0$

Case 2: $\underline{y = x} \xrightarrow{(1),(3),(4)} \begin{cases} z = 2\lambda x & \text{—(5)} \\ 2x = 2\lambda z & \text{—(6)} \\ 2x^2 + z^2 = 1 & \text{—(7)} \end{cases}$

Sub (5) in (6): $2x = 2\lambda(2\lambda x) \Rightarrow x = 2\lambda^2 x \Rightarrow x(1 - 2\lambda^2) = 0$

$\Rightarrow \underline{x = 0 \text{ or } \lambda = \pm \frac{1}{\sqrt{2}}}$

* $x = 0 \xrightarrow{(5)} z = 0$, $\underline{\text{rejected}}$ as $x = 0\ \&\ z = 0$ do not satisfy (7)

* $\lambda = \frac{1}{\sqrt{2}} \xrightarrow{(5)} z = \frac{2}{\sqrt{2}} x \Rightarrow \underline{z = \sqrt{2} x}$

$\xrightarrow{(7)} 2x^2 + 2x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$

$\boxed{\text{points}: (x,y,z) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)} \xrightarrow{f} \frac{\sqrt{2}}{2}$

* $\lambda = -\frac{1}{\sqrt{2}} \xrightarrow{(5)} z = -\frac{2}{\sqrt{2}} x \Rightarrow \underline{z = -\sqrt{2} x}$

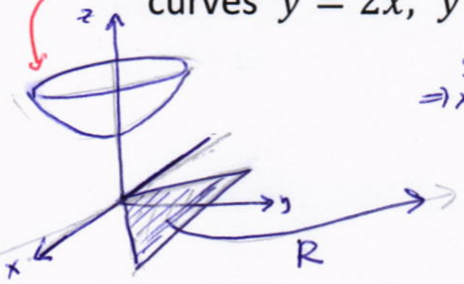
$\xrightarrow{(7)} 2x^2 + 2x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm \frac{1}{2}$

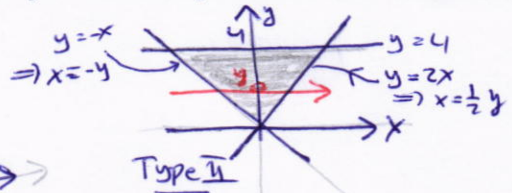
$\boxed{\text{points}: (x,y,z) = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2}\right)} \xrightarrow{f} -\frac{\sqrt{2}}{2}$

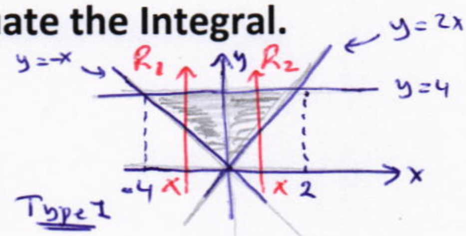
abs. max value of f is $\frac{\sqrt{2}}{2}$ & it occurs at $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$ & $\left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

abs min value of f is $-\frac{\sqrt{2}}{2}$ ——— $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ & $\left(\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$

3. [4 points] Set up an integral for the volume of the solid below the surface $z = 5 + x^2 + 2y^2$ and above the region R bounded by the curves $y = 2x,\ y = -x,\ y = 4$. **Do Not Evaluate the Integral.**







$R: -y \le x \le \frac{1}{2}y$

$0 \le y \le 4$

OR

$R_1: -4 \le x \le 0$

$-x \le y \le 4$

$R_2: 0 \le x \le 2$

$2x \le y \le 4$

$$V = \iint_R f(x,y)\, dA$$

$$= \underbrace{\int_0^4 \int_{-y}^{\frac{1}{2}y} (5 + x^2 + 2y^2)\, dx\, dy}_{3} \quad 1$$

$$\longrightarrow \text{OR} \iint_{R_1} f\, dA + \iint_{R_2} f\, dA$$

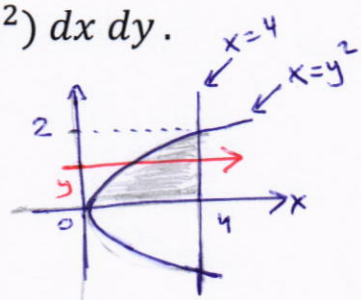
$$= \int_{-4}^{0} \int_{-x}^{4} (5 + x^2 + 2y^2)\, dy\, dx + \int_0^2 \int_{2x}^{4} (5 + x^2 + 2y^2)\, dy\, dx$$

4. [4 points] Evaluate $\int_0^2 \int_{y^2}^4 y \sin(x^2)\, dx\, dy$.

∘ Impossible to integrate. Swich:

$R: \quad y^2 \le x \le 4$

$0 \le y \le 2$



Type II

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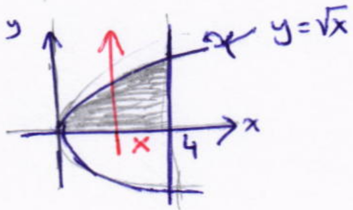
Type I

$R: \quad 0 \le x \le 4$

$0 \le y \le \sqrt{x}$



$$\int_0^2 \int_{y^2}^4 y \sin(x^2)\, dx\, dy = \int_0^4 \underbrace{\int_0^{\sqrt{x}} y \sin(x^2)\, dy}_{} \, dx \quad 2$$

$$\sin(x^2) \cdot \frac{1}{2} y^2 \Big]_{y=0}^{y=\sqrt{x}} \quad 0.5$$

$$= \int_0^4 \sin(x^2) \cdot \frac{1}{2} x \quad dx \quad 0.5$$

$$= -\frac{1}{4} \cos(x^2) \Big]_0^4 \quad 0.5$$

$$= -\frac{1}{4} \left(\cos(16) - 1\right) = \frac{1 - \cos(16)}{4} \quad 0.5$$