MATH 202.5 (Term 191)

Quiz 6 (Sects. 8.2 & 8.3) Duration: 20min

Name: ID number:

1.)(5pts)Solve the homogeneous linear system $X' = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix} X$.

2.)(5pts)Solve the system $X' = AX + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$, given that $\Phi(t) = \begin{pmatrix} e^{-t} & -2e^{2t} \\ e^{-t} & e^{2t} \end{pmatrix}$ is a fundamental matrix of $X' = AX$.

1.) $X' = AX, \quad A = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 \\ 1 & 6-\lambda \end{vmatrix} = (4-\lambda)(6-\lambda)+1$
$= \lambda^2 - 10\lambda + 25$
$= (\lambda - 5)^2$

$\lambda = 5$

$(A - 5I)K = 0 \; ; \; \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right), \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x + y = 0 \Rightarrow K \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t}$

$X_2 = (tK + P)e^{5t}, \quad (A - 5I)P = K$

$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right), \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$

$x + y = -1 \Rightarrow P \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\Rightarrow X_2 = \left[t\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{5t}$

$= \begin{pmatrix} t-1 \\ -t \end{pmatrix} e^{5t}$

$\Rightarrow \boxed{X = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} t-1 \\ -t \end{pmatrix} e^{5t},}$

$t \in (-\infty, \infty).$

2.) $\Phi^{-1} = \frac{1}{3e^{t}} \begin{pmatrix} e^{2t} & 2e^{2t} \\ -e^{-t} & e^{-t} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^{t} & 2e^{t} \\ -e^{-2t} & e^{-2t} \end{pmatrix}$

$\Phi^{-1}F = \frac{1}{3} \begin{pmatrix} 3te^{t} + 2 \\ -3te^{-2t} + e^{-3t} \end{pmatrix}$

$\int \Phi^{-1}F = \frac{1}{3} \begin{pmatrix} 3(t-1)e^{t} + 2t \\ (\frac{3}{2}t + \frac{3}{4})e^{-2t} - \frac{1}{3}e^{-3t} \end{pmatrix}$

$\Phi \int \Phi^{-1}F = \frac{1}{3} \begin{pmatrix} -\frac{9}{2} + (2t + \frac{2}{3})e^{-t} \\ \frac{9}{2}t - \frac{9}{4} + (2t - \frac{1}{3})e^{-t} \end{pmatrix}$

X_p

$X = \Phi C + X_p$

MATH 202.9 (Term 191)

Quiz 6 (Sects. 8.2 & 8.3) Duration: 20min

Name: **ID number:**

1.)(5pts)Solve the homogeneous linear system $X' = \begin{pmatrix} -6 & 1 \\ -2 & -4 \end{pmatrix} X$.

2.)(5pts)Solve the system $X' = AX + \begin{pmatrix} e^t \\ 2t \end{pmatrix}$, given that $\Phi(t) = \begin{pmatrix} e^{-2t} & 2e^t \\ e^{-2t} & 3e^t \end{pmatrix}$ is a fundamental matrix of $X' = AX$.

1.) $X' = AX$, $A = \begin{pmatrix} -6 & 1 \\ -2 & -4 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -6-\lambda & 1 \\ -2 & -4-\lambda \end{vmatrix} = (-6-\lambda)(-4-\lambda)+2 = \lambda^2+10\lambda+26$$

$\Delta = 100-104 = -4$

$\lambda = \frac{-10 \mp 2i}{2} = -5 \pm i$

$\lambda_1 = -5+i$ $\quad (A-(-5+i)I)K_1 = 0$

$$\left(\begin{array}{cc|c} -6-(-5+i) & 1 & 0 \\ -2 & -4-(-5+i) & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1-i & 1 & 0 \\ -2 & 1-i & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -\frac{(1-i)}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x - \frac{(1-i)}{2} y = 0$$

$K_1 \begin{pmatrix} 1-i \\ 2 \end{pmatrix}$

$B_1 = \operatorname{Re} K_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $B_2 = \operatorname{Im} K_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$X_1 = [B_1 \cos\beta t - B_2 \sin\beta t] e^{\alpha t}$, $\alpha = -5$, $\beta = 1$

$= \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos t - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t\right] e^{-5t}$

$X_2 = [B_2 \cos\beta t + B_1 \sin\beta t] e^{\alpha t}$

$= \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin t\right] e^{-5t}$

$$\Rightarrow \boxed{X = c_1 \begin{pmatrix} \cos t + \sin t \\ 2\cos t \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} -\cos t + \sin t \\ 2\sin t \end{pmatrix} e^{-5t}}$$

$t \in (-\infty, \infty)$.

2.) $\Phi^{-1} = \frac{1}{e^{-t}} \begin{pmatrix} 3e^t & -2e^t \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \begin{pmatrix} 3e^{2t} & -2e^{2t} \\ -e^{-t} & e^{-t} \end{pmatrix}$

$\Phi^{-1} F = \begin{pmatrix} 3e^{3t} - 4te^{2t} \\ -1 + 2te^{-t} \end{pmatrix}$

$\int \Phi^{-1} F = \begin{pmatrix} e^{3t} - (2t-1)e^{2t} \\ -t - 2(t+1)e^{-t} \end{pmatrix}$

$\underbrace{\Phi \int \Phi^{-1} F}_{X_p} = \begin{pmatrix} (1-2t)e^t \cancel{+6t+3} \\ (1-3t)e^t - 8t - 5 \end{pmatrix}$

$X = \Phi C + X_p$