1.) (5pts) Solve the homogeneous linear system 
\[ X' = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix} X. \]

2.) (5pts) Solve the system 
\[ X' = AX + \begin{pmatrix} 3t \\ -e^{-t} \end{pmatrix}, \]
given that \[ \Phi(t) = \begin{pmatrix} e^{-t} & -2e^{2t} \\ -e^{-t} & e^{2t} \end{pmatrix} \]
is a fundamental matrix of 
\[ X' = AX. \]

1.) 
\[ \begin{pmatrix} t ; \phi \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^t & 2e^t \\ -e^t & e^t \end{pmatrix} = \begin{pmatrix} e^t & 2e^t \\ -e^t & -2e^t \end{pmatrix}. \]

\[ \begin{pmatrix} \Phi^{-1} \\ \Phi F \end{pmatrix} = \begin{pmatrix} \frac{3}{2} t e^t + 2 \\ -3t e^{2t} + e^{3t} \end{pmatrix} \]

\[ \Phi \int \Phi F = \begin{pmatrix} \frac{3}{2} (t-1) e^t + 2e^t \\ (\frac{3}{2} t + \frac{3}{4}) e^{2t} - \frac{1}{3} e^{3t} \end{pmatrix} \]

\[ \Phi \int \Phi F = \begin{pmatrix} \frac{9}{2} t - \frac{9}{4} + (2t + \frac{2}{3}) e^{-t} \\ \frac{9}{2} t - \frac{9}{4} + (2t + \frac{2}{3}) e^{-t} \end{pmatrix} \]

\[ X \Phi \]

\[ X = \Phi C + X F \]
1. (5pts) Solve the homogeneous linear system $X' = \begin{pmatrix} -6 & 1 \\ -2 & -4 \end{pmatrix} X$.

2. (5pts) Solve the system $X' = AX + \begin{pmatrix} e^t \\ 2t \end{pmatrix}$, given that $\Phi(t) = \begin{pmatrix} e^{-2t} & 2e^t \\ e^{-2t} & 3e^t \end{pmatrix}$ is a fundamental matrix of $X' = AX$. 

$$\begin{align*}
1.) \quad X' &= AX, \quad A = \begin{pmatrix} -6 & 1 \\ -2 & -4 \end{pmatrix} \\
\det(A - \lambda I) &= \begin{vmatrix} -6 - \lambda & 1 \\ -2 & -4 - \lambda \end{vmatrix} = (-6 - \lambda)(-4 - \lambda) + 2 \\
\lambda^2 + 10\lambda + 26 &= \lambda = -5 \pm i \\
\chi &= -5 + i \\
\chi &= \frac{-10 \pm 2i}{2} = -5 \pm i \\
\lambda &= -5 + i \\
(\lambda - (-5 + i))K_1 &= 0 \\
\begin{pmatrix} -6 - (-5 + i) & 1 \\ -2 & -4 - (-5 + i) \end{pmatrix} \rightarrow \begin{pmatrix} -1 - i & 1 \\ -2 & 1 + i \end{pmatrix} \\
\begin{pmatrix} 1 + (-1 - i) \\ 0 \end{pmatrix} \chi - (i - 1)5y &= 0 \\
\begin{pmatrix} 1 - (-1 - i) \\ 0 \end{pmatrix} \\
K_1 &= \begin{pmatrix} 1 - (-1 - i) \\ 0 \end{pmatrix} \\
\beta_1 &= \text{Re} K_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \beta_2 = \text{Im} K_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
X_1 &= \begin{pmatrix} \beta_1 \cos \beta_1 - \beta_2 \sin \beta_1 \\ \beta_2 \cos \beta_1 + \beta_1 \sin \beta_1 \end{pmatrix} e^{\alpha t} \\
\alpha &= -5 \\
\beta &= 1 \\
X_1 &= \begin{pmatrix} (1) \cos t - (-1) \sin t \\ 0 \cos t + (1) \sin t \end{pmatrix} e^{-5t} \\
X_2 &= \begin{pmatrix} \beta_2 \cos \beta_1 + \beta_1 \sin \beta_1 \\ \beta_1 \cos \beta_1 - \beta_2 \sin \beta_1 \end{pmatrix} e^{\alpha t} \\
\alpha &= -5 \\
\beta &= 1 \\
X_2 &= \begin{pmatrix} (1) \cos t + (1) \sin t \\ -1 \cos t + (1) \sin t \end{pmatrix} e^{-5t} \\
X &= C_1 \begin{pmatrix} \cos t + (1) \sin t \\ 2 \cos t \end{pmatrix} e^{-5t} + C_2 \begin{pmatrix} -\cos t + 2 \sin t \\ -2 \sin t \end{pmatrix} e^{-5t} \\
&= \Phi(-5, -5) . \\

2.) \quad \phi = \frac{1}{e^t} \begin{pmatrix} 3e^t - 2e^{-t} \\ -e^t - 2e^{-t} \end{pmatrix} = \begin{pmatrix} 3e^t - 2e^{-t} \\ -e^t - 2e^{-t} \end{pmatrix} \\
\int \phi F &= \begin{pmatrix} 3e^t - 2e^{-t} \\ -e^t - 2e^{-t} \end{pmatrix} \\
\int \phi \phi F &= \begin{pmatrix} (3e^t - 2e^{-t})(3e^t - 2e^{-t}) \\ (-e^t - 2e^{-t})(-e^t - 2e^{-t}) \end{pmatrix} \\
&= \begin{pmatrix} (1-2t)e^t + 8t + 3 \\ (1-2t)e^t - 8t - 5 \end{pmatrix} \\
X &= \Phi C + \chi \phi \\
X &= \begin{pmatrix} 3e^t - 2e^{-t} \\ -e^t - 2e^{-t} \end{pmatrix}