Exercise 1. (10-5 points)
Let $V$ be an $n$-dimensional vector space over a field $K$ with a scalar product $(\cdot)$.  
(1) Prove that $V$ has an orthonormal basis.
(2) Set $V = \mathbb{C}^2$ as a vector space over $\mathbb{C}$ with the product scalar defined by:

$$(u|v) = x_1y_1 - i x_2 y_1 - i x_1 y_2 - 2 x_2 y_2, \ u = (x_1, x_2) \text{ and } v = (y_1, y_2).$$

Find an orthonormal basis of $V$.  

Exercise 2. (5-5-5-5 points)

Let $V$ be an $n$-dimensional vector space over a field $K$ with a non-degenerate scalar product $(\cdot, \cdot)$, and $V^*$ its dual space.

1. Prove that $\dim V = \dim V^*$.

2. Prove that for any linear functional $T \in V^*$, there is a unique $w \in V$ such that $Tv = (v|w)$ for every $v \in V$.

3. Set $V = \mathbb{R}^3$ with the standard scalar product, and let $B = \{v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (0,1,1)\}$ and $B^*$ its dual basis. Express $v_i^*(x, y, z)$ for each $i = 1, 2, 3$.

4. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by $T(x, y, z) = 3x + 5y - z$. Find $w \in \mathbb{R}^3$ such that $Tv = (v|w)$ for every $v \in \mathbb{R}^3$. 
Exercise 3. (5-5)

(1) Find the bilinear form associated to the quadratic form $q(x, y, z) = x^2 + 2xy + 2xz$.

(2) Find the quadratic form associated to the bilinear form:

$f(X, Y) = x_1y_1 + x_1y_2 + x_1y_3 + x_2y_1 + x_2y_3 + x_3y_1$. 
Exercise 4. (8-7 points)

Let $V = \mathbb{R}^3$ with the scalar product defined by:

$$(X|Y) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2.$$ 

(1) Find an orthogonal basis of $V$ with respect to the defined scalar product.

(2) Find the index of positivity.
Exercise 5. (5-10)
Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ with a non-degenerate scalar product $(\cdot)$.
Let $u, v$ fixed vectors of $V$, and $T$ the map defined by $Tw = (u|w)v$.

(1) Show that $T$ is a linear operator on $V$.

(2) Find the transpose $^tT$ of $T$. 

Exercise 6. (5-5-5)
Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ with a positive definite Hermitian product $(\cdot)$. Let $T$ be a linear operator on $V$ such that $TT^* = T^*T$.

(1) Show that $(Tu|Tv) = (T^*u|T^*v)$ for every $u, v$ in $V$.

(2) Let $B = \{v_1, \ldots, v_n\}$ be a basis of $V$ and $L : V \to V$ defined by $L(v_i) = \sum_{j=i}^{n} c_{ij} v_j$. Find the matrix $A$ representing $L$ in the basis $B$ ($A = [L]_B$).

(3) Show that if $L = L^*$, then $A$ is diagonal.
Exercise 7. (5-5)
Let $A$ be an $n \times n$ real unitary, upper triangular matrix.

(1) Prove that if $n = 3$, then $A$ is a diagonal matrix.

(2) Is this true for all positive integers $n$? Justify.