Exercise 1. (5-5-5-5 points)

Let $V$ be an $n$-dimensional vector space over the real field $\mathbb{R}$ with a positive definite scalar product $(\cdot)$ and let $T$ be a linear operator on $V$ such that $T^2 = T$ and $tTt = T^tT$.

(1) Prove that $V = \ker(T) \oplus \text{Im}(T)$.

(2) Prove that $\ker(T) = \ker(T^t)$.

(3) Prove that for every $v \in V$, $v = w + T(u)$ where $w \in \ker(T)$, $\|T(v)\|^2 = \|T(u)\|^2 = (T(v)|T(v))$.

(4) Prove that $T = tTt$. 
Exercise 2. (5-5-5-5 points)

Let $V$ be an $n$-dimensional vector space over a field $K$ and $N$ a nilpotent operator on $V$ (i.e. $N^r = 0$ for some positive integer $r \leq n$).

(1) Prove that the only eigenvalue $\lambda$ of $N$ is $\lambda = 0$.

(2) Let $c$ be a scalar ($c \in K$) and $T = N + cI$. Find all eigenvalues and the characteristic polynomial of $T$.

(3) Assume that $V = \mathbb{R}^3$ and $T(x, y, z) = (4x + y, 4y + z, 4z)$. Write $T = N + cI$ where $N$ is a nilpotent operator and $c$ a constant to be determined.

(4) Without any calculations, find the eigenvalues of $T$ and its characteristic polynomial.
Exercise 3. (5-5-5-5-5)

Label each of the following statements as TRUE or FALSE and justify your answer.

Let $V$ be an $n$-dimensional real vector space with a definite positive scalar product.

(1) If $T : V \to V$ is a linear operator on $V$, then neither $\ker(T)$ nor $\text{Im}(T)$ are necessarily $T$-invariant.

(2) If $A$ and $B$ are linear operator on $V$ such that $AB = BA$, then $A$ and $B$ have the same eigenvectors.

(3) For any linear operator on $V$ with characteristic polynomial $f$, $f$ is a polynomial with minimal degree satisfying $f(T) = 0$.

(4) A linear operator $T$ is normal if $TT^* = T^*T$. Every normal operator is diagonalizable.

(5) Any $n \times n$ matrix with complex coefficients is triangulable.
Exercise 4. (5-5-5-5 points)
Let $V$ be an $n$-dimensional vector space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$, or $\mathbb{K} = \mathbb{R}$) with a positive definite (hermitian) product $(\langle \cdot, \cdot \rangle)$.

1. Assume that $\mathbb{K} = \mathbb{C}$ and let $A$ be a linear operator on $V$ such that $\langle Av, v \rangle = 0$ for all $v \in V$. Prove that $A = 0$.

Now, assume that $\mathbb{K} = \mathbb{R}$ and let $A$ be a linear operator on $V$ such that $\langle Av, v \rangle = 0$ for all $v \in V$.

2. Prove that $A + A^t = 0$.

3. Prove that if $A$ is symmetric, then $A = 0$.

4. Find a linear operator $A$ on $V$ such that $\langle Av, v \rangle = 0$ for all $v \in V$ but $A \neq 0$. 
Exercise 5.  (5-5-5-5)
Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ with a positive definite hermitian product $\langle \cdot \rangle$.

(1) Let $T$ be a hermitian linear operator on $V$. Prove that the eigenvalues of $T$ are real.

(2) Let $A$ be an invertible linear operator on $V$ and $P = A^*A$. Show that $P$ is a hermitian operator and the eigenvalues of $P$ are positive.

(3) Show that there is a linear operator $B$ on $V$ such that $P = B^2$.

(4) Show that there is a unitary operator $U$ on $V$ such that $A = UB$. 

Exercise 6. (7-5-8)
Let $V$ be an $n$-dimensional vector space over the real field $\mathbb{R}$ with a positive definite scalar product $(\cdot)$ and let $T$ be a unitary operator on $V$.

(1) Prove that $V$ has a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_r$ where $V_i$ is $T$-invariant, $V_i \perp V_j$ and $\dim V_i = 1, 2$.

(2) Assume that $V = \mathbb{R}^4$, $S$ its standard basis and $T$ is the linear operator given by $T(x, y, z, t) = (t, y, z, x)$. Verify that $T$ is a unitary operator.

(3) Find a direct sum decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ of $V$. 
Exercise 7. (8-6-6)
Let $V = \mathbb{C}^3$ as a vector space over $\mathbb{C}$, $S$ its standard basis and $T$ the linear operator given by $T(x, y, z) = (2x + iz, iy, ix)$.

(1) Find a fan of $T$.

(2) Find a basis $B$ of $V$ where the matrix $[T]_B$ representing $T$ is upper triangular.

(3) Find an invertible matrix $P$ such that $P^{-1}[T]_SP$ is upper triangular.
Exercise 8. (10-5)

Let $V$ be an $n$-dimensional vector space over the complex field $\mathbb{C}$, $T$ a linear operator on $V$ and $f$ its characteristic polynomial.

(1) Prove that $f(T) = 0$.

(2) Assume that $T$ is invertible and $f = a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + X^n$. Find the characteristic polynomial of $T^{-1}$. 