Exercise 1 (7 points)
Let \((u_n)\) be sequence of real numbers. Define the sequence \((v_n)\) by

\[ v_n = \frac{u_1 + u_2 + \ldots + u_n}{n} \]

1. (5 points) Show that if the sequence \((u_n)\) is convergent and \(\lim(u_n) = l\), then \((v_n)\) is convergent and \(\lim(v_n) = l\).

2. (2 points) Find a sequence \((u_n)\) such that \((v_n)\) is convergent and \((u_n)\) is divergent.
Exercise 2 (7 points)

Let \((u_n)\) be a sequence of real numbers defined by \(u_0 = \frac{3}{2}\) and \(u_{n+1} = (u_n - 1)^2 + 1\)

1. (2 pts) Prove that for each \(n \in \mathbb{N}\), \(1 < u_n < 2\)

2. (2 pts) Prove that the sequence \((u_n)\) is strictly monotone.

3. (3 pts) Deduce that \((u_n)\) is convergent and compute its limit.
Exercise 3 (7 points)
Let \( (u_n) \) be a bounded sequence of real numbers. Let us define
\[ v_n = \sup \{ u_k; k \geq n \} \quad \text{and} \quad w_n = \inf \{ u_k; k \geq n \} \]

1. (2 pts) Show that the sequence \( (v_n) \) is decreasing and \( (w_n) \) is increasing.

2. (2 pts) Deduce that \( (v_n) \) and \( (w_n) \) are convergent sequences.

3. (3 pts) Prove that the sequence \( (u_n) \) is convergent if and only if \( \lim (v_n) = \lim (w_n) \).
Exercise 4 (7 points)
Let $a$ and $b$ two real numbers such that $a < b$ and $f : [a, b] \rightarrow [a, b]$.

1. (4 pts) Suppose that for every $x, y$ in $[a, b] : |f(x) - f(y)| \leq |x - y|$. Show that $f$ is continuous. Deduce that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

2. (3 pts) Suppose that for every $x, y$ such that $x \neq y$ we have $|f(x) - f(y)| < |x - y|$, then there exists one and only one $x_0 \in [a, b]$ such that $f(x_0) = x_0$. 

Exercise 5 (7 points)

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that for every \( x, y \) in \( \mathbb{R} \): \( f(x + y) = f(x) + f(y) \).

1. (1 pts) Compute \( f(0) \) and show that \( f(-x) = -f(x) \).

2. (2 pts) Prove that for every \( x \in \mathbb{R} \) and \( n \in \mathbb{Z} \): \( f(nx) = nf(x) \).

3. (2 pts) Prove that for every \( x \in \mathbb{R} \) and \( q \) rational: \( f(qx) = qf(x) \).

4. (2 pts) Prove that for every \( x \in \mathbb{R} \) and \( \lambda \) real: \( f(\lambda x) = \lambda f(x) \).
Exercise 1: We have \( \forall a \in A: a \leq \sup(A) \) and \( \forall b \in B: b \leq \sup(B) \)

\[
\Rightarrow a + b \leq \sup(A) + \sup(B), \quad \forall a, b.
\]

Then \( \sup(A) + \sup(B) \) is an upper bound of \( A + B \).

Since \( \sup(A + B) \) is the smallest upper bound of \( A + B \) then:

\[
\sup(A + B) \leq \sup(A) + \sup(B). \quad (1) \quad (1.5 \text{pt})
\]

Conversely, \( \sup(A + B) \) is an upper bound of \( A + B \) then \( \forall a, b: a + b \leq \sup(A + B) \)

\[
\Rightarrow a \leq \sup(A + B) - b, \quad \forall a, b.
\]

Then \( \sup(A + B) - b \) is an upper bound of \( A \)

\[
\Rightarrow \sup(A) \leq \sup(A + B) - b, \quad \forall b \in B.
\]

\[
\Rightarrow b \leq \sup(A + B) - \sup(A), \quad \forall b \in B.
\]

\[
\Rightarrow \sup(A + B) - \sup(A) \text{ is an upper bound of } B
\]

\[
\Rightarrow \sup(B) \leq \sup(A + B) - \sup(A)
\]

\[ \Leftrightarrow \quad \sup(A) + \sup(B) \leq \sup(A + B). \quad (2) \quad (1.5 \text{pt}) \]

(1) and (2) \( \Rightarrow \quad \sup(A) + \sup(B) = \sup(A + B) \).
20/ It is clear that \( (u_{2k}) = (2^{2k}) \) is an increasing and unbounded. \( \Rightarrow \) \( \lim (u_{2k}) = +\infty \) \( \Rightarrow \) A has no supremum. \( \boxed{15 \text{ pt}} \)

\( (u_{2k+1}) \) is \( (2^{2k+1}) \), this sequence has positive terms and \( \lim (u_{2k+1}) = 0 \), then \( \inf (A) = 0 \). \( \boxed{15 \text{ pt}} \)

Exercise 2.1: \( u_0 = \frac{3}{2} \), then \( 1 < u_0 < 2 \), the property is true = prove \( n = 0 \).

Suppose that \( 1 < u_n < 2 \), let us show that \( 1 < u_{n+1} < 2 \).

\( 1 < u_n < 2 \) \( \Rightarrow \) \( 0 < u_{n-1} < 1 \) \( \Rightarrow \) \( 0 < (u_{n-1})^2 < 1 \),

\( \Rightarrow \) \( 1 < (u_{n-1})^2 + 1 < 2 \) \( \Rightarrow \) \( 1 < u_{n+1} < 2 \).

2. Let \( n \in \mathbb{N} \), \( u_{n+1} - u_n = (u_{n-1})^2 + 1 - u_n \)

\( = u_n^2 - 3u_n + 2 \)

\( = (u_n - 2)(u_n - 1) < 0 \)

because \( u_n < 2 \),

\( \Rightarrow \) \( u_{n+1} < u_n \) \( \Rightarrow \) \( (u_n) \) is strictly decreasing.

3. \( (u_n) \) is monotone and bounded \( \Rightarrow \) \( (u_n) \) is convergent.

let \( \lambda = \lim (u_n) = \lim (u_{n+1}) \) \( \Rightarrow \lambda = \lambda - 1 \)

\( \boxed{3 \text{ pt}} \), \( \Rightarrow \boxed{\lambda = 1} \).
**Ex. 8.**

All \((a_n)\) is decreasing because the sequences of

\[ A_n = \{ a_k \mid k \geq n \} \]

is decreasing.

- Idea: for \((a_n)\).

21/ \((v_n)\) is bounded, then \((v_n)\) and \((w_n)\) are bounded. Then \((w_n)\) and \((v_n)\) are bounded and monotone, then they are convergent.

31/ It is clear that \(\forall n \in \mathbb{N}\),

\[ w_m \leq u_n \leq v_m. \]

If \(\lim (v_n) = \lim (w_m)\), then by squeeze \(w_m\)
\((v_n)\) is convergent and \(\lim (v_n) = \lim (w_m)\). Suppose that \((v_n)\) is convergent. and \(\lim (v_n) = l\).

By definition of \(\sup (u_k; k \geq m)\),

\[ \forall \epsilon > 0; \exists N \in \mathbb{N}; \forall m \geq N; \quad |u_m - l| < \epsilon \]

\[ \forall m \geq N; \quad m \geq N, \forall k \geq m; \quad l - \epsilon < u_k < l + \epsilon. \]

\[ \forall k / \forall \eta > m \] is bounded from below by \(l - \epsilon\)
and bounded above by \(l + \epsilon\). Then:

\[ l - \epsilon < w_m \leq v_m < l + \epsilon. \]

\[ \Rightarrow (w_n) \) and \((v_n)\) converge to \(l.\)
Exercise 4: \( f: [a,b] \rightarrow [a,b] \), \( x_0 \in [a,b] \).

Let \( \varepsilon > 0 \); \( \exists \delta > 0 \): \( \left| x - x_0 \right| < \delta \Rightarrow \left| f(x) - f(x_0) \right| < \varepsilon \).

We know that \( \left| f(x) - f(x_0) \right| \leq \left| x - x_0 \right| . \leq \varepsilon \)

if \( \left| x - x_0 \right| < \varepsilon \).

Then \( \delta = \varepsilon \).

\(:= \) \( f \) is continuous at \( x_0 \) (for all \( x_0 \in [a,b] \)).

Consider \( g(x) = f(x) - x \).

We know that \( f(x) \in [a,b] \), then \( f(a) \geq a \)

\( \Rightarrow \) \( g(a) \geq 0 \).

\( f(b) \in [a,b] \Rightarrow f(b) \leq b \Rightarrow f(b) - b \leq 0 \)

\( \Rightarrow \) \( g(b) = 0 \).

Apply the intermediate value theorem to find \( c \in [a,b] \)

such that : \( g(c) = 0 \Rightarrow f(c) = c \).

\( \Rightarrow \) If \( \left| f(x) - f(y) \right| < \left| x - y \right| \Rightarrow \left| f(x) - f(y) \right| < \left| x - y \right| \)

apply the result of question 1 to deduce existence

of \( c \in [a,b] \): \( f(c) = c \).

Let us prove that this element \( c \) is unique.

Suppose there exist \( x_1 \) and \( x_2 \in [a,b] \):

\( f(x_1) = x_1 \) and \( f(x_2) = x_2 \).
We have \( |f(x) - f(y)| < |x - y| \)
but we have \( |f(x) - f(y)| = |x - y| \)

\[ \implies |x - y| < |x - y| \,
\]
which is a contradiction.

Exercise 5: Let \( f : \mathbb{R} \to \mathbb{R} \) be \( f(x+y) = f(x) + f(y) \).

1. \( f(0) = f(0+0) = f(0) + f(0) \implies f(0) = 0 \)

2. \( f(nx) = f(x+x+\ldots+x) = f(x)+\ldots+f(x) \)
by induction \( \implies f(nx) = nf(x) \).

Suppose true for \( n = \tau (n+1) \).

In fact even if \( \tau \in \mathbb{Z} \) is negative, we have \( f(-x) = -f(x) \).

Suppose \( \tau \) is negative; \( \tau = -(m) \).

\[ f(-nx) = f(-((-m)x)) = -f((m)x) = -(-m)f(x) = mf(x) \]

3. \( \text{Let } \frac{p}{q} \in \mathbb{Q} \quad (p, q) \in \mathbb{Z} \times \mathbb{Z} \)

\[ f(x) = f(q \cdot (\frac{1}{q}x)) = q \cdot f(\frac{1}{q}x) \]

\[ \implies f(\frac{p}{q}x) = \frac{1}{q} f(\frac{1}{q}x) \]

Thus \( f(\frac{p}{q}x) = f(\frac{p}{q} \cdot \frac{1}{q}x) = \frac{p}{q} f(\frac{1}{q}x) \)
4. Take \( \lambda \in \mathbb{R} \), \( f(q_n) \subseteq \mathbb{Q} \) s.t.:

\[
\lim_{n \to \infty} q_n = \lambda,
\]

\[
f(x) = f\left(\lim_{n \to \infty} q_n x\right) = \lim_{n \to \infty} f(q_n x) \quad (f \text{ is continuous}),
\]

\[
= \lim_{n \to \infty} q_n \cdot f(q_n) = \lambda \cdot f(\mu).
\]