Exercise 1  Suppose that $a > 0$ and that $f$ is Riemann integrable on $[-a, a]$ that is $f \in \mathcal{R}([-a, a])$

1) If $f$ is even show that $\int_{-a}^{a} f(x)dx = 2 \int_{0}^{a} f(x)dx$

2) If $f$ is odd show that $\int_{-a}^{a} f(x)dx = 0$.

3) If $f$ is continuous on $[-a, a]$, show that $\int_{-a}^{a} f(x^2)dx = 2 \int_{0}^{a} f(x^2)dx$.

Exercise 2.

1) Show that if $f : [a, b] \rightarrow R$ is continuous then it is Riemann integrable on $[a, b]$.

2) Show that if $f : [a, b] \rightarrow R$ is monotone then it is Riemann integrable on $[a, b]$.

Exercise 3

1) Suppose that $f$ is bounded on $[a, b]$ and that there exist two sequences of tagged partitions $\mathcal{P}_n$ and $\mathcal{R}_n$ of $[a, b]$ such that $\lim \|\mathcal{P}_n\| = \lim \|\mathcal{R}_n\| = 0$ and that $\lim S(f, \mathcal{P}_n) \neq \lim S(f, \mathcal{R}_n)$. Show that $f$ is not Riemann integrable.

2) Consider the Dirichlet function defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that $f$ is not Riemann integrable.

Exercise 4.

1) let $f, g : [a, b] \rightarrow R$ Riemann integrable functions, such that $f(x) \leq g(x)$ for every $x$ in $[a, b]$.

Show that

$$\int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx$$

2) Consider the following functions $f : [0, 1] \rightarrow R$, such that $f(x) = \sin(1/x)$ on $[0, 1]$ and $f(0) = 0$; $g(x) = \frac{x}{x^2 + 1}$ on $[9, 12]$; $h(x) = \exp(x^2)$ on $[-3, 6]$, $k : [0, 1] \rightarrow R$, such that $k(x) = \ln(x)$ on $[0, 1]$ and $k(0) = 0$. Which one of these functions is Riemann integrable and which one is not (justify your answer with simple arguments without detailed proofs).
Exercise 1: \[ f(x) = 0 = x_1 < x_2 < \ldots < x_n = a \] is a partition of \([0,a]\). This class that -\( x_1 = -a < \ldots < -x_n = 0 \) is a partition of \([-a,0]\).

Let \( x_1^*, x_2^*, \ldots , x_n^* \) be tagged points of the partition \([0,a]\).

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{m} f(x_i^*) \Delta x_i + \sum_{i=1}^{m} f(x_i^*) \Delta x_i \]

\[ = \lim_{n \to \infty} \sum_{i=1}^{m} \left( f(x_i^*) + f(x_i^*) \right) \Delta x_i \]

\[ = 2 \lim_{n \to \infty} \sum_{i=1}^{m} f(x_i^*) \Delta x_i \]

all \( g(x) = f(x^2) \); it is clear that \( g \) is even, then \( \int_{a}^{b} f(x^2) \, dx = 2 \int_{0}^{b} f(x) \, dx \).
Exercise 2: \( f \) is not Riemann integrable.

Exercise 3: Suppose that \( f \) is Riemann integrable, then there exists \( \delta > 0 \) such that for every \( \Phi_n \) such that \( \| \Phi_n \| < \delta \), we have
\[ \lim_{n \to \infty} S(f, \Phi_n) = L. \]

Then if \( \hat{\Phi}_n \) is another tagged partition such that
\[ \lim_{n \to \infty} \| \hat{\Phi}_n \| = 0 \Rightarrow \lim_{n \to \infty} S(f, \hat{\Phi}_n) = L. \]

This is a contradiction with the hypotheses.

Then \( f \) is not Riemann integrable.

\[ f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise} \end{cases} \]
Consider a tagged partition $P_m$ with rational tags. (This is always possible because $\mathbb{Q} \cap [0,1]$ is dense in $[0,1]$).

5. $S(f; P_m) = 1$.

Consider another tagged partition $P_m$ with irrational tags. (same argument!)

6. $S(f; P_m) = 0$.

Then \( \lim S(f; P_m) = 1 \) and \( \lim S(f; P_m) = 0 \).

Thus \( f \) is not $R$. integrable.

\[ \]

Example 4: Let $\Phi$ be a tagged partition and $f$:

\[ \lim ||P_m|| = 0. \]

\[ S(f; P_m) = \sum_{i=1}^{m} f(x_i^*) \Delta x_i \leq \sum_{i=1}^{m} g(x_i) \Delta x_i = \int_a^b g(x) \, dx \]

7. \( \lim S(f; P_m) \leq \lim S(g; P_m) \)

8. \( \int_a^b f \, dx \leq \int_a^b g \, dx \)
All of $f$ is $R$-integrable because

is bounded and continuous except in $0$.

$g$ is $R$-int. because it is continuous on $[0,12] - [-3,6]$.

$h$ is not $R$-integrable because it is not bounded.